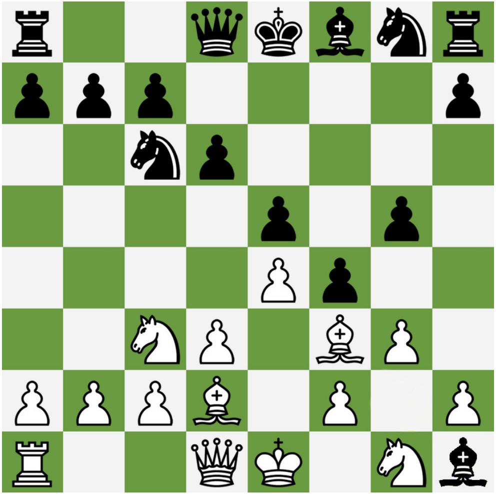
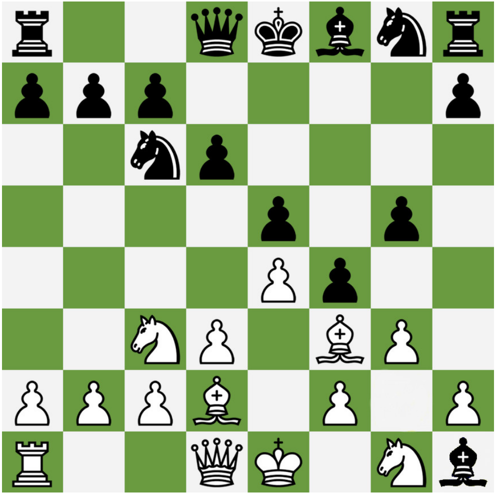
A characterization of half-positional ω-regular languages

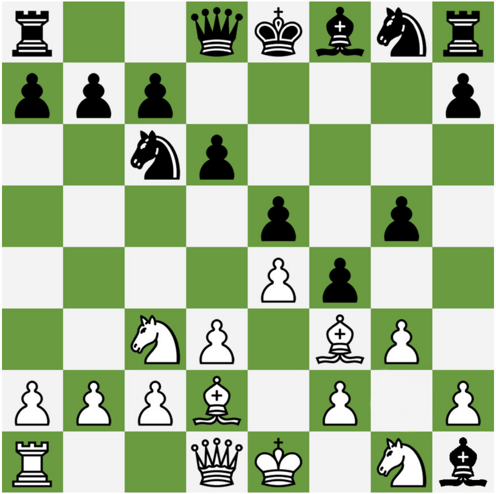
Antonio Casares, Pierre Ohlmann

Automata Seminar IRIF
13 October 2023





White to move.
What is the optimal move?



Does not depend on the past of the play**

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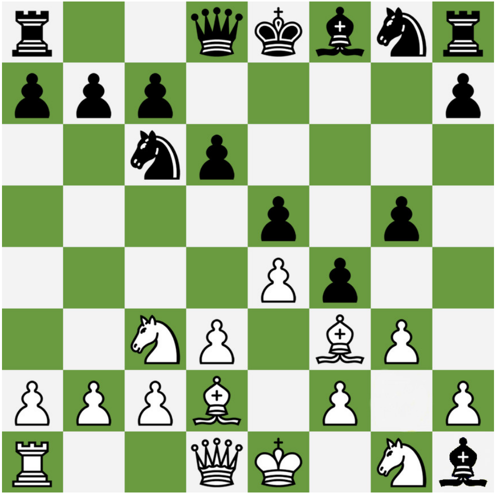

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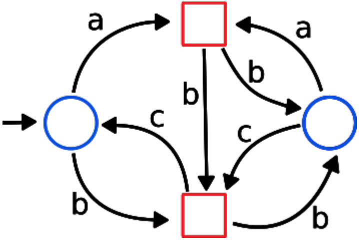
Games on Graphs



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- Players move a token in turns producing an infinite word $w \in \Sigma^\omega$.

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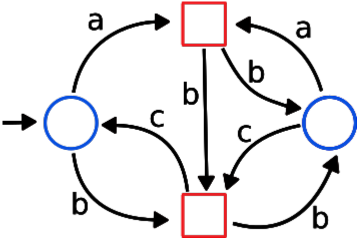

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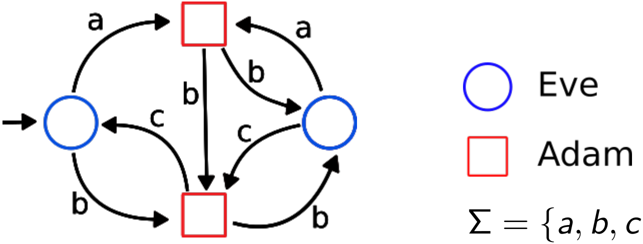

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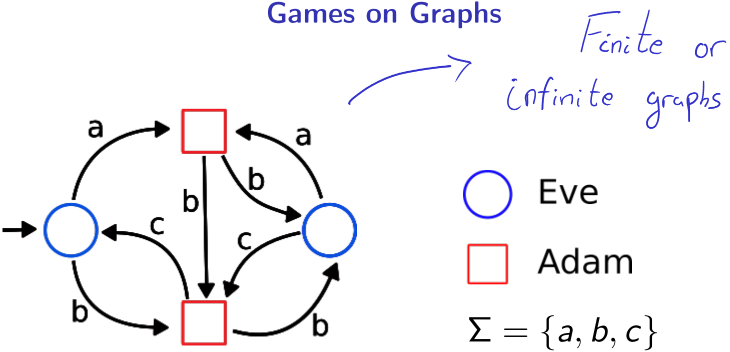

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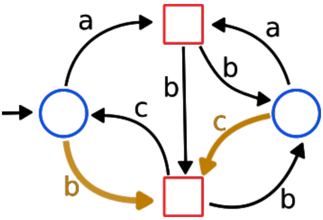
Positional Strategies



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$$\sigma\colon V_{\text{Eve}} \to E.$$

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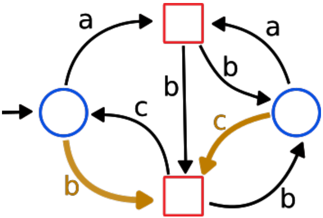

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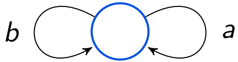
The positional strategy above ensures the language

$$L = \text{ Words containing the factor } bc \text{ infinitely often} = \mathtt{Inf}(bc).$$

Example: Positional strategies do not always suffice



$$L = \texttt{Inf}(a) \wedge \texttt{Inf}(b)$$

Eve wins, but not positionally.

Positional Languages

Half-positionality

A language L is **half-positional** if, for every game $\mathcal{G}$ using L as winning condition:

Eve can win $\mathcal{G}$ $\implies$ She can win using a positional strategy.

Positional Languages

over finite/arbitrary games

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Bipositionality

A language L is **bipositional** if both L and $\Sigma^\omega \setminus L$ are half-positional.

ω-regular languages

The class of ω-**regular languages** can be defined equivalently as those that are:

- Recognizable by a non-deterministic Büchi automaton.
- Recognizable by a deterministic parity automaton.
- Definable by ω-regular expressions.
- Definable in MSO with successor.
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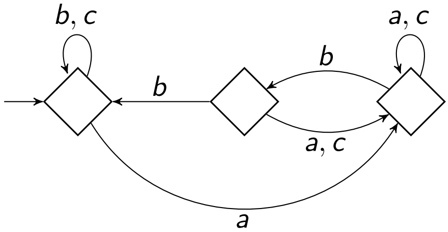

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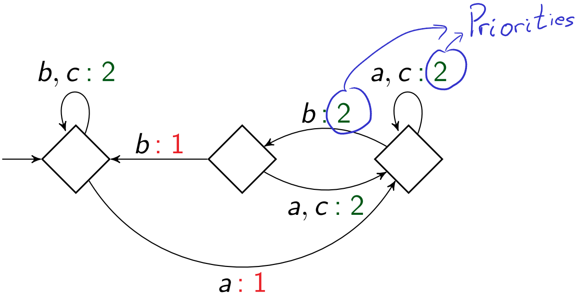

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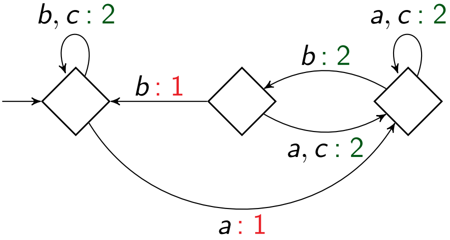

Run accepting if
min{*priorities seen infinitely often*} is even.

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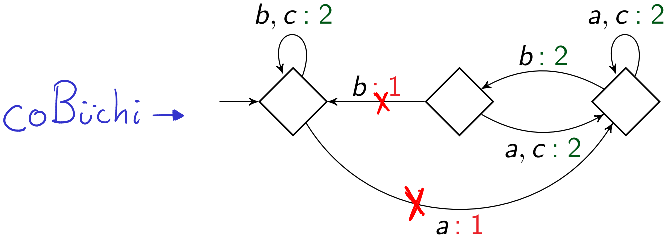

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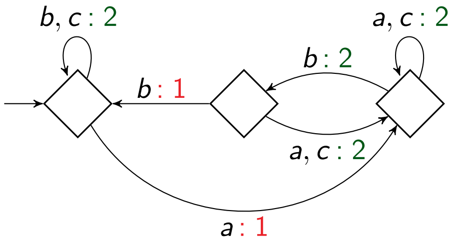

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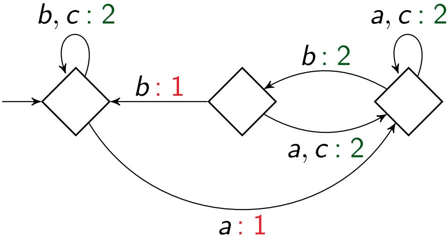

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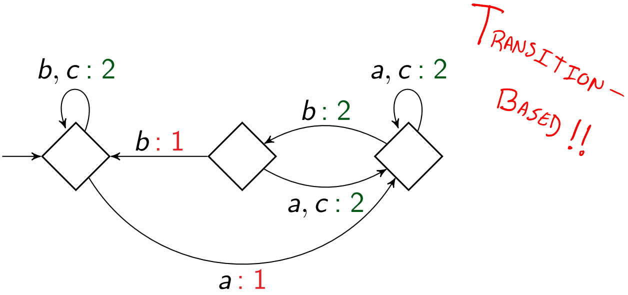

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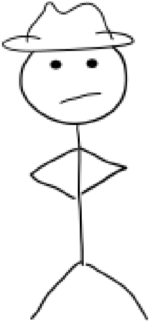
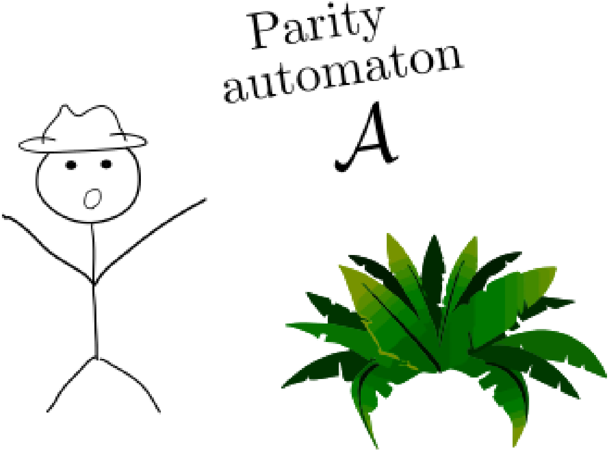
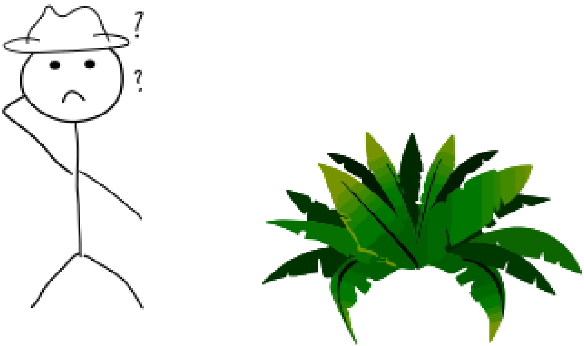
Theoretical Computer Scientist in the Wild




Parity
automaton
$\mathcal{A}$

Is $\mathcal{L}(A)$ half / bi positional?



What do we already know?

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→ 1-to-2 player lift

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Not much...[*]

[*] This is of course not true. There are many very interesting results about half-positionality.

What do we know about half-positionality?

Decidability for finite graphs (Kopczyński 2007)

The following problem is decidable:

Input:	DPA $\mathcal{A}$ recognising a prefix-independent language.
Question:	$\mathcal{L}(\mathcal{A})$ half-positional over finite graphs?

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L is half-positional over all graphs

iff

L admits well-ordered monotone universal graphs.

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NOT CONSTRUCTIVE!

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Always $\Uparrow$ iff $\Downarrow$ If L admits a neutral letter

L admits well-ordered monotone universal graphs.

Conjecture: L always admits a neutral letter

Questions about half-positionality (for ω-regular languages)

Decidability for infinite graphs?

Can we decide whether L is half-positional (all graphs)?
Complexity?

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1-to-2-player lift?

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False for non ω-regular languages

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Questions About Half Positionality

Kopczyński's Conjecture (2006)

Let L_1 and L_2 be two prefix-independent half-positional languages. Then:

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- Does not hold for finite graphs (Kozachinskiy, 2022).

  Non ω-regular counterexample!

- Open for infinite graphs.

Contributions

Main Contribution

Characterization of half-positional ω-regular languages

We identify a class $\mathcal{C}$ of deterministic parity automata such that:

- $\mathcal{A} \in \mathcal{C} \implies \mathcal{L}(\mathcal{A})$ is half-positional.
- L half-positional $\implies L$ can be recognized by some $\mathcal{A} \in \mathcal{C}$.

Corollaries of our Characterization

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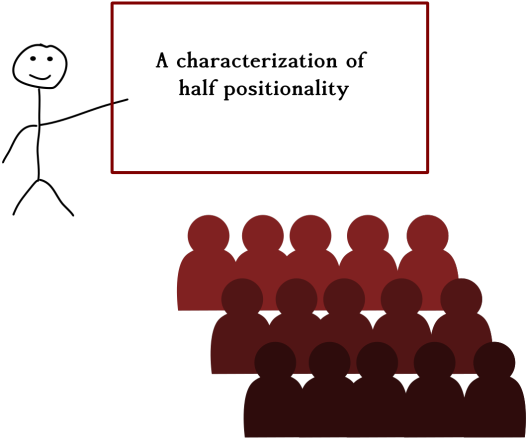
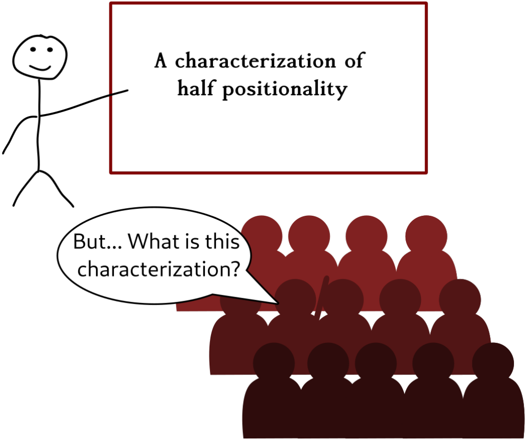
Ohlmann's Neutral Letter Conjecture for ω-regular languages

For L an ω-regular language:

$$L \text{ half-positional} \iff (L + \textit{neutral letter}) \text{ is half-positional.}$$


A characterization of half positionality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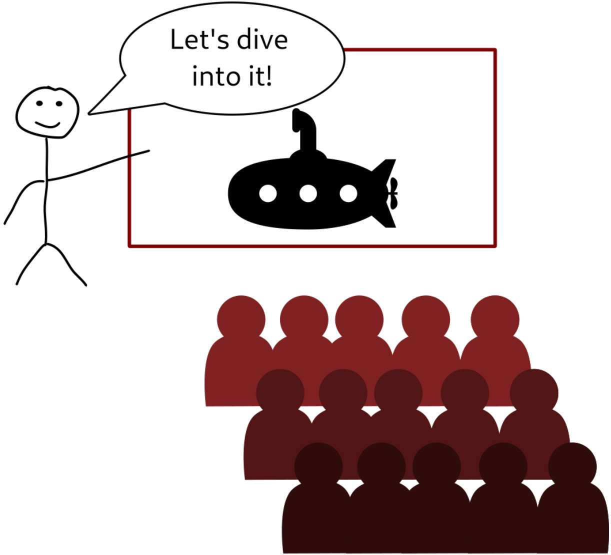
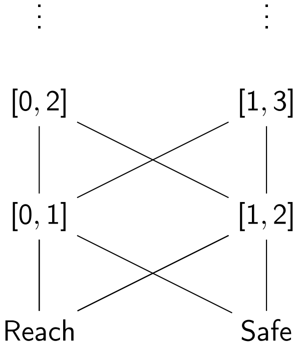
But... What is this
characterization?


Let's dive into it!

Towards the characterization

A hierarchy of languages



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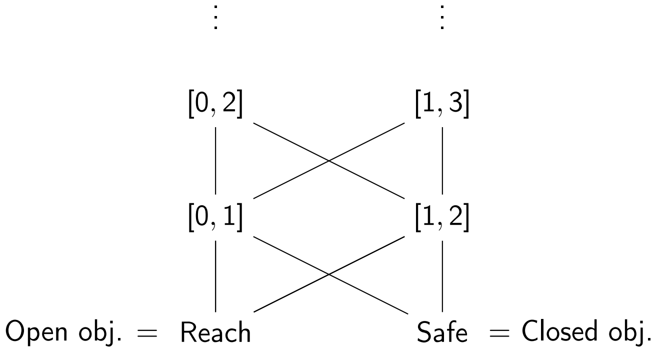

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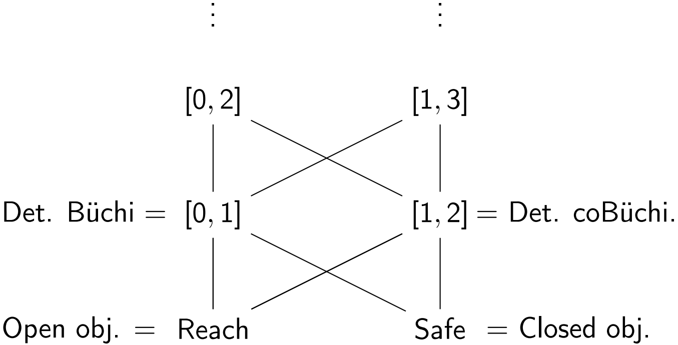

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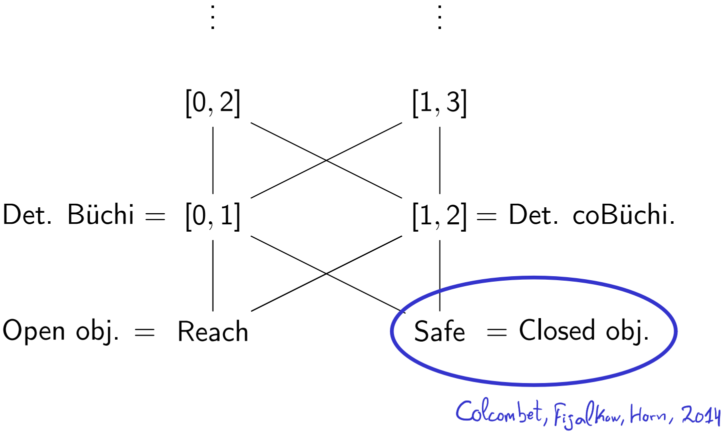

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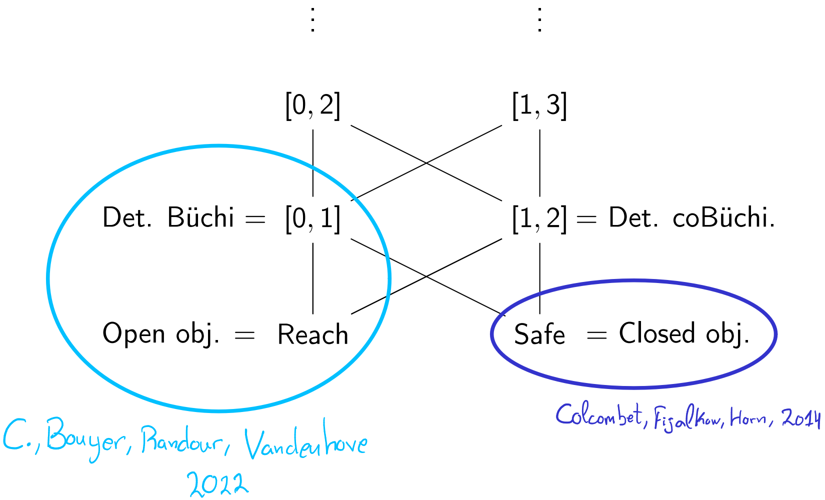

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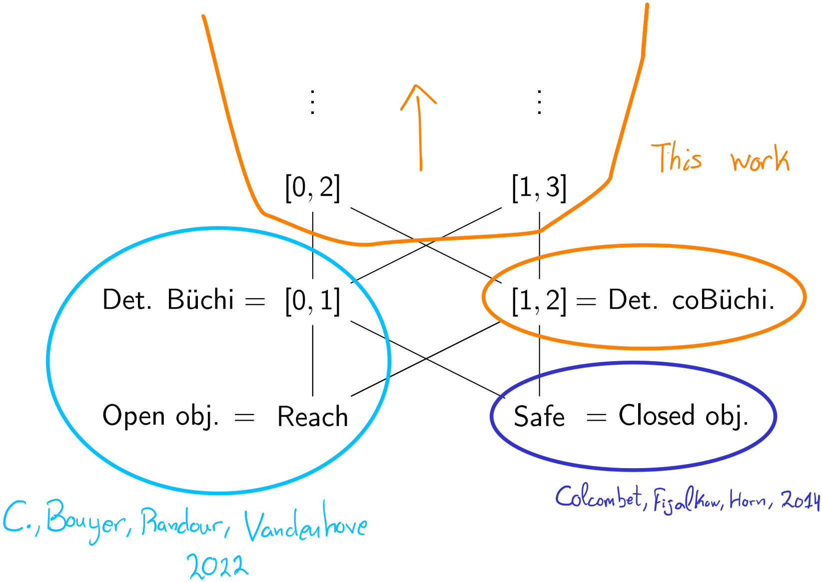

But before climbing the ladder...

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How to prove a characterization of positionality?

$\mathcal{C}$ a class of languages.

Goal: L half-positional iff $L \in \mathcal{C}$.

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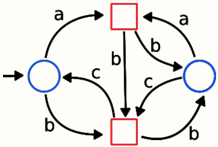

Show game in which Eve can win, but not using a positional strategy.

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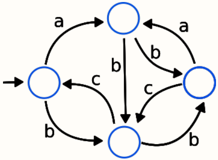

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  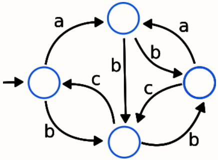

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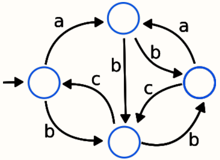

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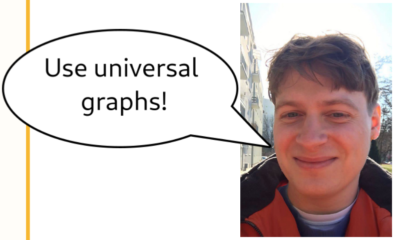

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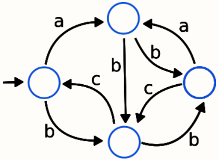


Show **finite 1-player** game in which Eve can win, but not using a positional strategy.

$\exists$ Well-ordered monotone L-universal graph

$\Downarrow$

L is half-positional

$\mathcal{C}$ a class of languages.

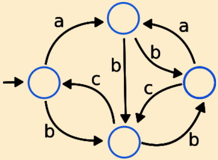
Goal: L half-positional iff $L \in \mathcal{C}$.

- L half-positional **over finite 1-player graphs** $\implies L \in \mathcal{C}$

- $L \in \mathcal{C} \implies L$ half-positional

Assume $L \notin \mathcal{C}$

This talk



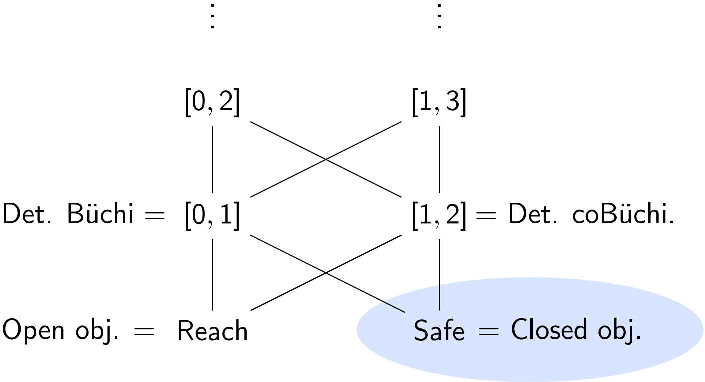
Show **finite 1-player** game in which Eve can win, but not using a positional strategy.

$\exists$ Well-ordered monotone L-universal graph

$\Downarrow$

L is half-positional

Closed objectives



Residuals

We fix an language $L \subseteq \Sigma^\omega$.

For a finite word $u \in \Sigma^*$ we write

$$u^{-1}L = \{w \in \Sigma^\omega \mid uw \in L\}.$$

$$\mathrm{Res}(L) = \text{ Residuals of } L.$$

➔ We consider Res(L) **ordered by inclusion**.

Closed objectives: Total order of residuals

Proposition

If $L \subseteq \Sigma^\omega$ is half-positional, then $\mathrm{Res}(L)$ *is totally ordered by inclusion.*
If L is topologically closed, this condition is also sufficient.

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Proof (necessity):
On the contrary, there are $u_1, u_2 \in \Sigma^*$ and $w_1, w_2 \in \Sigma^\omega$ such that:

$$u_1 w_1 \in L, \quad u_2 w_1 \notin L,$$

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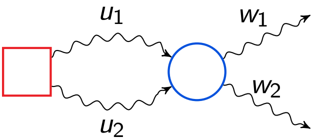

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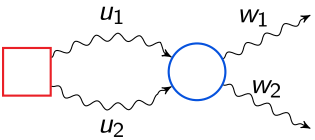
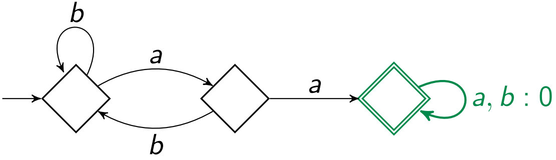


We can make it
- finite
- 1-player

Example: Total order not sufficient in general

$$L = \{w \in \Sigma^\omega \mid w \text{ contains the factor } aa\}.$$



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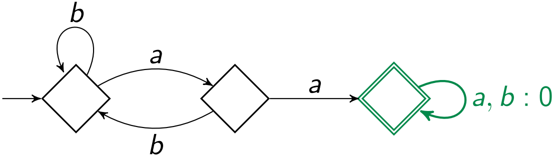

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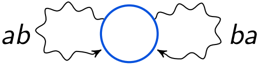

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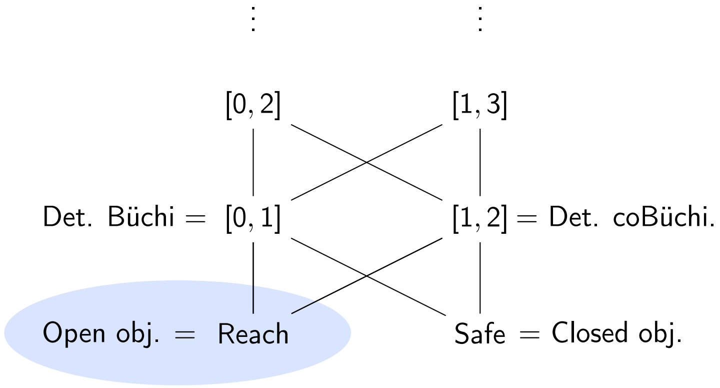

Progress consistency

Language L is ***progress consistent*** if for all $u, v \in \Sigma^*$:

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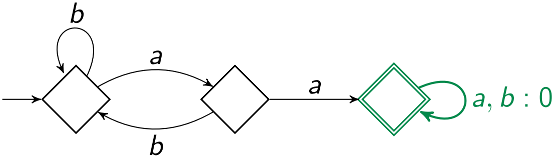

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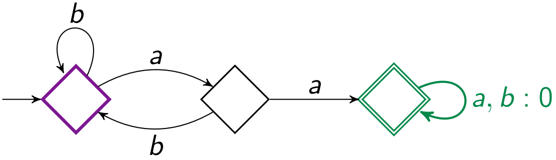

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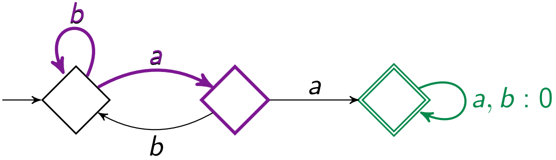

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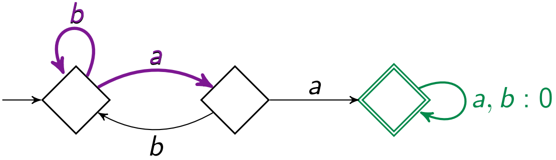

$$\varepsilon^{-1}L \quad \subsetneq \quad a^{-1}L \quad \subsetneq \quad aa^{-1}L = \Sigma^\omega$$

$u = \varepsilon,\ \ v = ba$, but $(ba)^\omega \notin L \implies L$ not progress consistent.

Open objectives: Progress consistency + Total order

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If L is half-positional, then it is progress consistent.

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Let u, v such that $u^{-1}L \subsetneq uv^{-1}L$ but $uv^\omega \notin L$.
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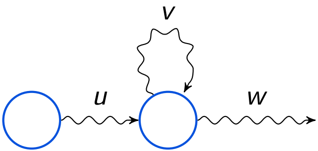

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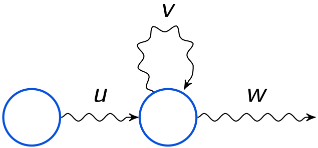
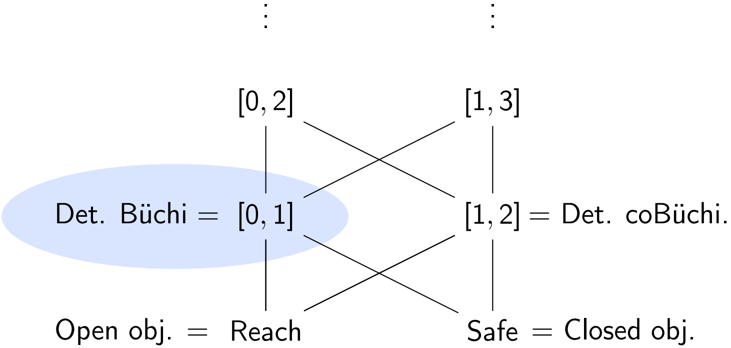


Proposition

If L is topologically open, it is half-positional if and only if:

1. *Total order of residuals, and*
2. *Progress consistent.*

Objectives recognized by deterministic Büchi automata



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Let L be recognizable by a deterministic Büchi automaton. L is half-positional if and only if:

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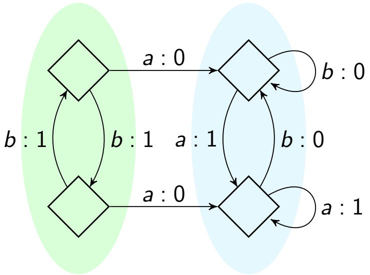

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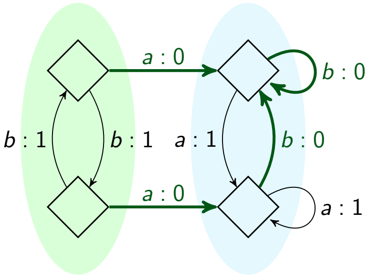

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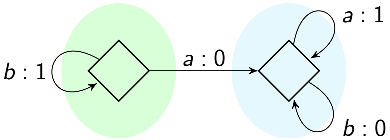
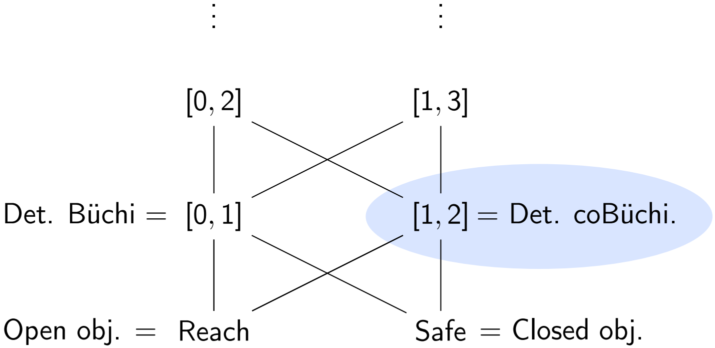
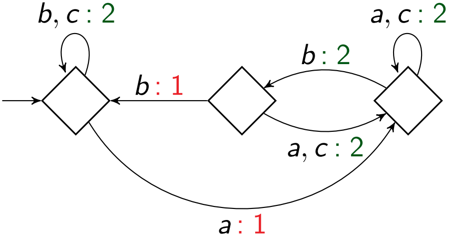


Objectives recognized by deterministic coBüchi automata



Example: Objective recognized by coBüchi automaton



$$\mathcal{L}(\mathcal{A}) = \texttt{Fin}(a) \vee \texttt{Fin}(bb)$$

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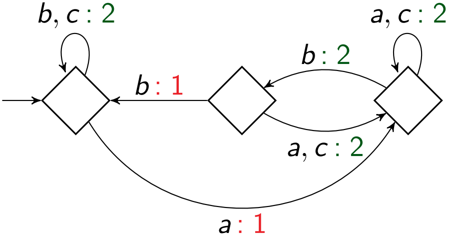

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- Only one residual (prefix-independent)
- Claim: Half-positional

Safe languages and safe components

For a state q we write:

$\mathcal{L}_{\mathsf{safe}}(q) = \{w \in \Sigma^* \mid$ the path $q \xrightarrow{w}$ does not produce $1\}$.

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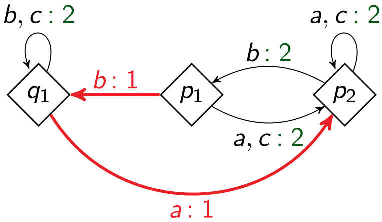

- $\mathcal{L}_{\mathsf{safe}}(q_1) = (b + c)^*$
- $\mathcal{L}_{\mathsf{safe}}(p_1) = (a + c)\overline{(\Sigma^* bb\Sigma^*)}$
- $\mathcal{L}_{\mathsf{safe}}(p_2) = \overline{(\Sigma^* bb\Sigma^*)}$

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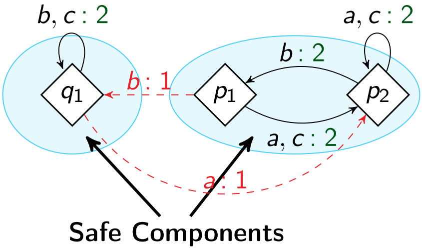

CoBüchi: Total order in safe components

Claim

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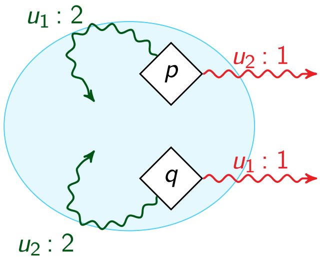

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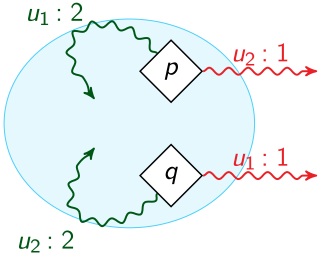

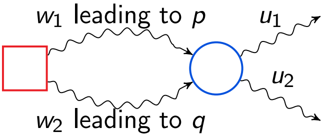

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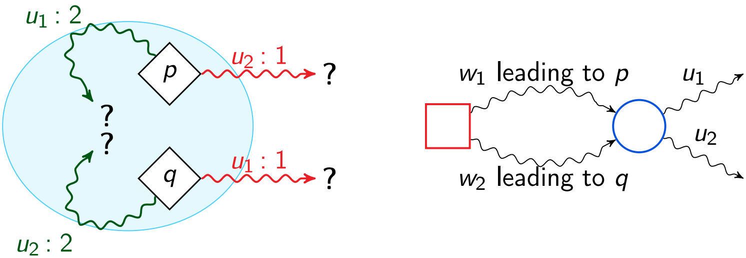
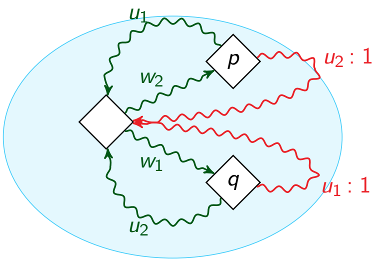
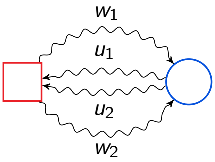


And now what? We need to iterate this process.

Synchronizing words

Objective: Obtain words giving the following situation





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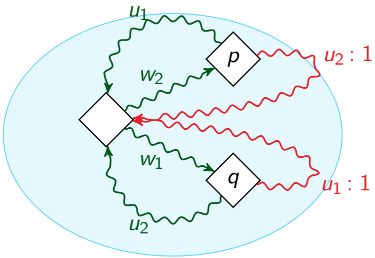

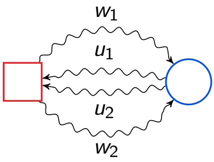

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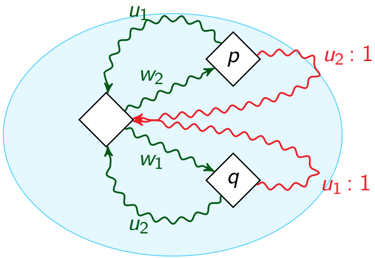
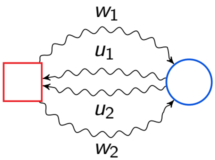


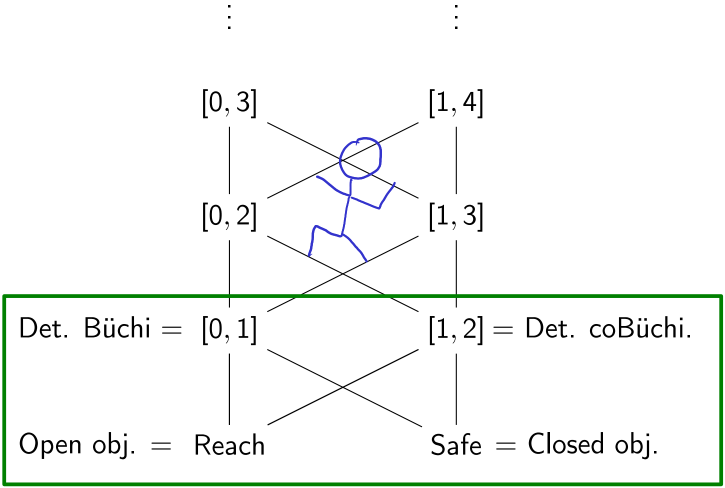


- An arbitrary automaton does not admit synchronizing words.
- But we can obtain a **minimal history-deterministic*** automaton which always admits synchronizing words!†

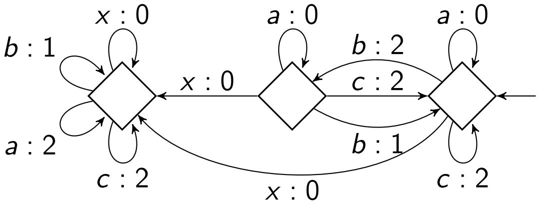
*History-deterministic = Good-for games

†Minimization method from Abu Radi and Kupferman "*Minimizing GFG Transition-Based Automata*" ICALP 2019.

Climbing the ladder



Proof by example



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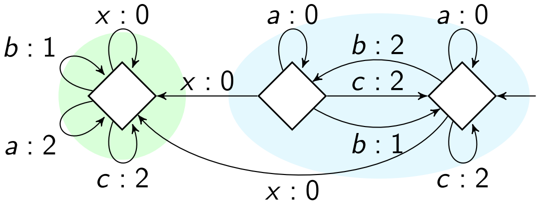

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- Residuals totally ordered ✓
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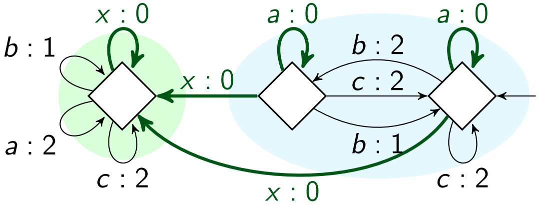

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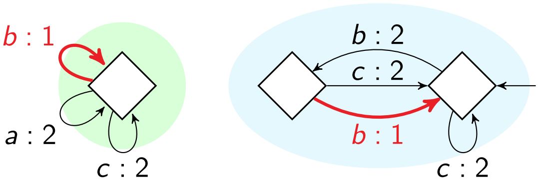

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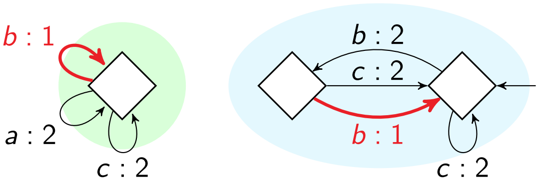

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L half-positional $\iff$ L can be recognized by a parity automaton admitting a decomposition by layers

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- Formal details quite technical

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Input: Deterministic parity automaton $\mathcal{A}$.

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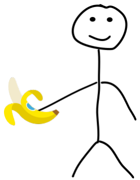

You can peel it

$\Downarrow$

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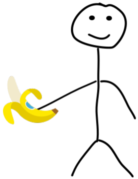

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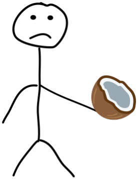

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It is a coconut

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Thanks for your attention!