Memory requirements for generalised reachability games

Antonio Casares

Generalised reachability conditions

Let Σ be an alphabet and $L \subseteq \Sigma^*$ a (regular) language of finite words.

$Reach(L) = \{w \in \Sigma^\omega \ : \ \exists u \in \Sigma^\omega$ a prefix of w such that $u \in L\}$

We will suppose that L is suffix-closed (we can always take $L \cdot \Sigma$).

Generalised reachability conditions are exactly topologically open conditions (without regularity assumptions).

Problem

Given a language L, what is the optimal memory required by the existential player (Eve) to win games with condition $Reach(L)$?

That is, what is the number M_L such that:

- If Eve wins a game with condition $Reach(L)$, she has an strategy using memory of size at most M_L.
- For every L there exists a $Reach(L)$-game won by Eve where she requires at least M_L memory states to win.

Upper bound

If L is a regular language, let $\mathcal{A}$ be the minimal deterministic automaton for it. Then, given a $Reach(L)$-game $\mathcal{G}$ won by Eve, the game $\mathcal{G} \times \mathcal{A}$ is won by her positionally.

Conclusion

The automaton $\mathcal{A}$ can be used as a memory structure for every $Reach(L)$-game and $M_L \leq |\mathcal{A}|$.

Examples

$\Sigma = \{a, b\}$

Example 1: $L = aaa \cdot \Sigma^*$

Left quotients: $\varepsilon^{-1}L \subseteq a^{-1}L \subseteq aa^{-1}L \subseteq aaa^{-1}L$.

For every L-game, 1 memory state suffices (we meet the lower bound).

Example 2: $L = aa^*b \cdot \Sigma^*$

Left quotients: $\varepsilon^{-1}L \subseteq a^{-1}L \subseteq ab^{-1}L$.

The lower bound we provide is 1, but we need 2 memory states for a game with one vertex controlled by Eve.

Lower bound

For $w \in \Sigma^*$, its *left quotient* is

$$w^{-1}L = \{u \in \Sigma^* \ : \ wu \in L\}$$

The set of left quotients is naturally ordered by inclusion.

An *antichain* is a subset A of a partially ordered set such that every pair of elements of it are incomparable.

In [CFH14] the memory requirements for the universal player are established (i.e. topologically closed conditions):

Theorem [CFH14]

The memory required by the universal player is given by the size of a maximal antichain of left quotients of $L \cdot \Sigma^*$. (No regularity assumptions).

This lower bound holds for the existential player too, but it is not tight.

Lower bound

M_L is at least as big as a maximal antichain of left quotients of $L \cdot \Sigma^*$.

References

[CFH14] Thomas Colcombet, Nathanaël Fijalkow, and F. Horn. "Playing Safe". In: *FSTTCS*. 2014.

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