Half-Positional Objectives Recognized by Deterministic Büchi Automata

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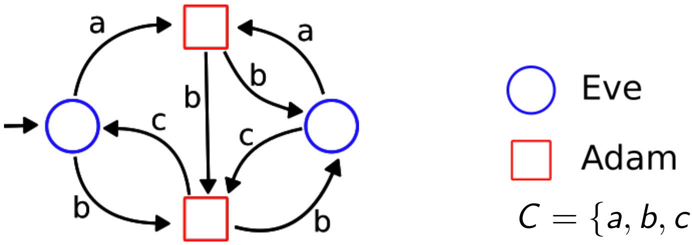
Games and Positionality

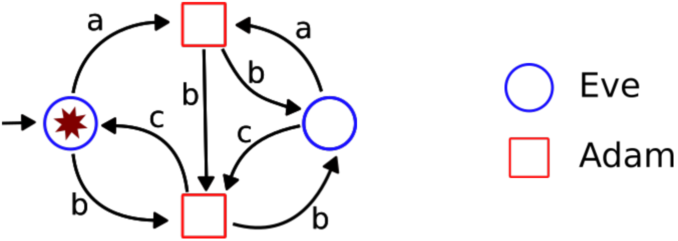
Games over graphs

Arena: oriented graph $\mathcal{G} = (V = (V_{\mathrm{Eve}} \uplus V_{\mathrm{Adam}}), E, v_0)$ with edges labeled by colors in a set C and an initial vertex.



$C = \{a, b, c\}$

Players move a token in turns producing an infinite word $w \in C^\omega$.



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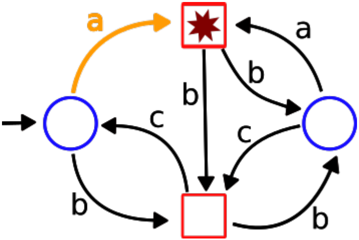
$\text{Output} =$

Games over graphs



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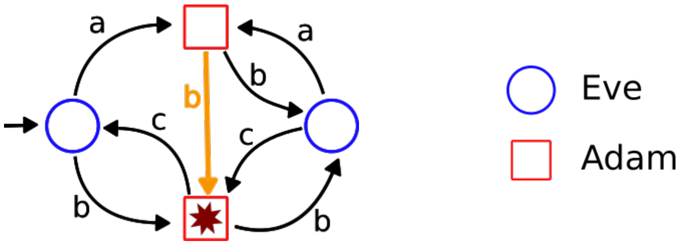
Games over graphs



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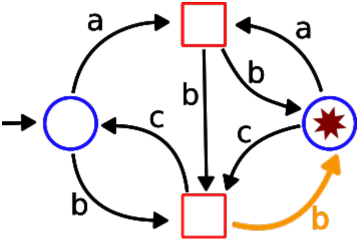
$$\mathrm{Output} = ab$$



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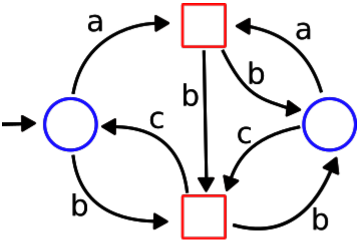


Objective: Set $\mathbb{W} \subseteq C^\omega$ of winning sequences.

Eve wins a play if $w \in \mathbb{W}$.

Adam wins a play if $w \notin \mathbb{W}$.

$\mathbb{W}$-game = Arena + Objective $\mathbb{W}$

Strategy (for Eve)

Function $\sigma : E^* \times V_{\mathrm{Eve}} \to E$ prescribing how Eve should move depending on the past of the play.

Strategies

Strategy (for Eve)

Function $\sigma : E^* \times V_{\mathrm{Eve}} \to E$ prescribing how Eve should move depending on the past of the play.

Winner

We say that Eve *wins a* $\mathbb{W}$*-game* $\mathcal{G}$ if she has a strategy σ such that all paths from v_0 consistent with that strategy belong to $\mathbb{W}$.

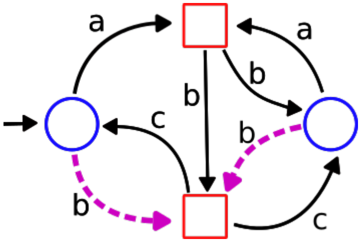
Positional Strategies

Positional strategy

Function

$$\sigma : V_{\mathrm{Eve}} \to E,$$

(Eve's choices depend exclusively on the current position).



$$\mathbb{W} = (b + c)^{\omega}$$

Positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is *half-positional*[1] if for every $\mathbb{W}$-game $\mathcal{G}$ Eve has a positional strategy σ such that

$$\text{Eve wins } \mathcal{G} \implies \text{Eve wins } \mathcal{G} \text{ using } \sigma.$$

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Positionality

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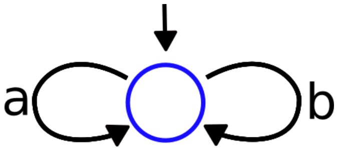
Bi-positional objective

An objective $\mathbb{W} \subseteq C^\omega$ is *bi-positional* if both $\mathbb{W}$ and $\mathcal{C}^\omega \setminus \mathbb{W}$ are half-positional.

[1] In this talk positional = half-positional.

Examples

- $\mathbb{W} = C^*(ab)^\omega$ is not positional.



Examples

- $\mathbb{W} = C^* a^4 C^\omega$

?

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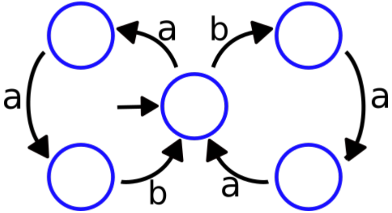
Not positional



Examples

- $\mathbb{W} = \mathrm{Inf}(a) = (Ca)^\omega$.

Positional (Emerson, Jutla 1991)

Examples

- $\mathbb{W} = C^* a^2 C^\omega \cup \mathrm{Inf}(a)$.

?

Examples

- $\mathbb{W} = C^* a^2 C^\omega \cup \mathrm{Inf}(a)$.

You will know at the end of the talk!

Some known results

Bi-positionality is quite well understood:

Some known results about bi-positionality

- Characterization of bi-positionality over finite arenas [Gimbert, Zielonka '05].
- Characterization of bi-positionality over all arenas [Colcombet, Niwiński '06].

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But for applications in synthesis, half-positionality is more relevant!

Some known results

Some known results about half-positionality

- Some sufficient conditions for half-positionality [Kopczyński '08, BFMM '10].
- Charact. of half-positional objectives over all arenas using universal graphs [Ohlmann '21].

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- **Characterization of half-positional objectives over all arenas using universal graphs [Ohlmann '21].**

➔Structural characterization:

$\mathbb{W}$ positional $\Leftrightarrow$ There exists a suitable structure for any cardinal.

Not effective.

Contribution

Open question

Effective characterization of positionality for ω-regular objectives.

In this work:

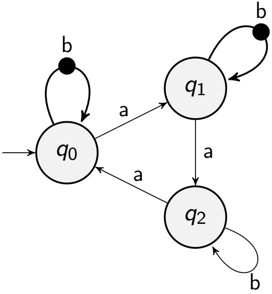
Main result

Effective characterization of positionality for languages recognized by **deterministic Büchi automata**.

Deterministic Büchi Automata

Büchi automata

In this talk, all automata will be deterministic.



- Automaton $\mathcal{B}$
- $C = \{a, b\}$ input alphabet
- Input $= w \in C^\omega$
-  = Büchi transition

➔ We accept a run if it visits infinitely often a Büchi transition.

➔ $\mathcal{L}(\mathcal{B}) = \{w \in C^\omega \;:\; \mathcal{B}$ has an accepting run over $w\}$.

Remark: Acceptance condition is defined over *transitions* of the automata.

DBA-recognizability

We say that an objective $\mathbb{W} \subseteq C^\omega$ is **DBA-recognizable** if there is a deterministic Büchi automaton $\mathcal{B}$ such that

$$\mathbb{W} = \mathcal{L}(\mathcal{B}).$$

Remark: the class of DBA-recognizable objectives is a proper subclass of ω-regular objectives.

$$\omega\mathrm{-regular} = \text{ Recognizable by ND-Büchi } = \text{ Recognizable by det. parity.}$$

Right congruence

We fix an objective $\mathbb{W} \subseteq C^\omega$.

For a finite word $u \in C^*$ we write

$$u^{-1}\mathbb{W} = \{w \in C^\omega \ : \ uw \in \mathbb{W}\}.$$

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For $u, v \in C^*$:

$$u \prec v \quad \text{if} \quad u^{-1}\mathbb{W} \subsetneq v^{-1}\mathbb{W} \quad \text{(Preorder).}$$

$$u \sim v \quad \text{if} \quad u^{-1}\mathbb{W} = v^{-1}\mathbb{W} \quad \text{(Equivalence relation).}$$

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→ Analogous relations between the states of a DBA $\mathcal{B}$.

Right congruence

On finite words:

One state per equivalence class.

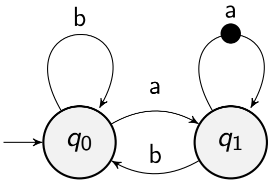
Right congruence

On finite words:

One state per equivalence class.

On infinite words:

This is not always possible!



$$\mathcal{L}(\mathcal{B}) = (C^* aa)^\omega$$

Only one equivalence class.

Myhill-Nerode-like objectives

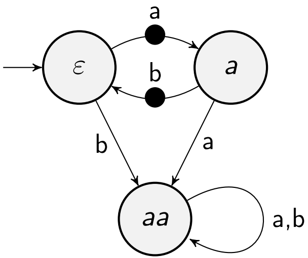
Myhill-Nerode-like objective

If an objective $\mathbb{W}$ can be recognized by a DBA with one state per equivalence class we say that it is **Myhill-Nerode-like**.

Remember: Transition based acceptance!

Example

- $\mathbb{W} = (ab)^\omega$



- $aa \prec \varepsilon$
- $aa \prec a$
- ε and a are incomparable.

Three sufficient and necessary conditions for half-positionality

Condition 1: $\prec$ is a total order

Condition 1

Prefix preorder $\prec$ is total.

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Lemma (Necessity of Condition 1)

If $\prec$ is not total, $\mathbb{W}$ is not positional.

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Proof: If $\prec$ is not total, there are $u_1, u_2 \in C^*$ and $w_1, w_2 \in C^\omega$ such that:

$$u_1 w_1 \in \mathbb{W}, \quad u_2 w_1 \notin \mathbb{W},$$

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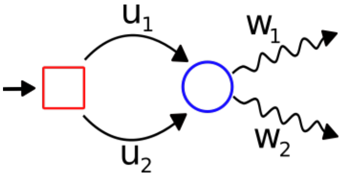
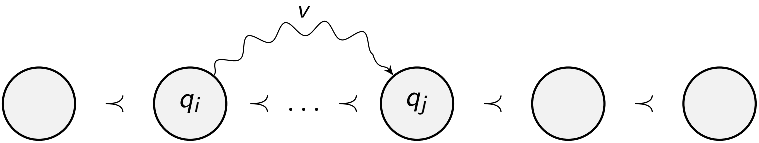


Condition 2: Progress-consistency

Condition 2

We say that $\mathbb{W}$ is *progress-consistent* if for all $u, v \in C^*$:

$$u \prec uv \implies uv^\omega \in \mathbb{W}.$$



Then, v^ω is accepted if read from q_i.

Condition 2: Progress-consistency

Lemma (Necessity of Condition 2)

If $\mathbb{W}$ is not progress-consistent, it is is not positional.

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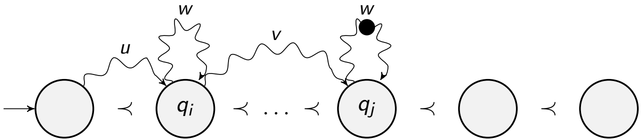
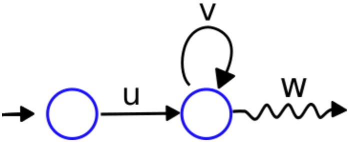
If $\mathbb{W}$ is not progress-consistent, it is is not positional.

Proof: Let u, v such that $u \prec uv$ and $uv^\omega \notin \mathbb{W}$. There is $w \in C^\omega$ s.t. $uw \notin \mathbb{W}$ but $uvw \in \mathbb{W}$.





Condition 3: Recognizability by the prefix-classifier

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Objective $\mathbb{W}$ is Myhill-Nerode-like (one state per equivalence class).

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Condition 3: Recognizability by the prefix-classifier

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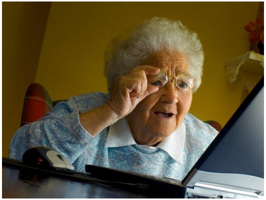
Objective $\mathbb{W}$ is Myhill-Nerode-like (one state per equivalence class).

Lemma (Necessity of Condition 3)

If $\mathbb{W}$ is not Myhill-Nerode-like, it is not positional.

Proof: Quite technical.



Necessity of the conditions

Conditions for half-positionality

$\mathbb{W} \subseteq C^\omega$ a DBA-recognizable objective.

- Prefix preorder $\preceq$ is total.
- Progress-consistency.
- Myhill-Nerode-like.

Proposition (Necessity of the conditions)

If a DBA-recognizable objective $\mathbb{W} \subseteq C^\omega$ is half-positional , then it verifies the three previous conditions.

Sufficiency of the conditions

Conditions for half-positionality

$\mathbb{W} \subseteq C^\omega$ a DBA-recognizable objective.

- Prefix preorder $\preceq$ is total.
- Progress-consistency.
- Myhill-Nerode-like.

They are also sufficient!

Sufficiency of the conditions

The main ingredient to prove the sufficiency of the conditions is:

Theorem (Ohlmann 2021)

*An objective $\mathbb{W} \subseteq C^\omega$ is half-positional if and only if for every cardinal κ there exists a $(\mathbb{W}, \kappa)$-**universal well-monotonic graph** $\mathcal{U}$.*

The existence of such graphs is a structural witness of positionality.

Sufficiency of the conditions

Proposition

If $\mathbb{W} \subseteq C^\omega$ *is a DBA-recognizable objective verifying*

- *Prefix preorder* $\preceq$ *is total,*
- *Progress-consistency,*
- *Being Myhill-Nerode-like,*

then, there is a $(\mathbb{W}, \kappa)$*-universal well-monotonic graph for every cardinal* κ*.*

Main result

Conditions for half-positionality

$\mathbb{W} \subseteq C^\omega$ a DBA-recognizable objective.

- Prefix preorder $\preceq$ is total.
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- Myhill-Nerode-like.

Theorem

A DBA-recognizable objective $\mathbb{W} \subseteq C^\omega$ is half-positional if and only if it verifies the three previous conditions.

Corollaries

Corollary (Complexity)

Given a Büchi automaton $\mathcal{B}$, we can determine in polynomial time whether $\mathcal{L}(\mathcal{B})$ is half-positional.

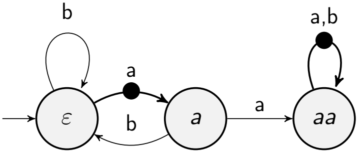
Corollary (1-to-2 players lift)

Let $\mathbb{W}$ be a DBA-recognizable objective. If $\mathbb{W}$ is positional ***over finite one-player arenas****, then it is half-positional* ***over all arenas*** *(2 players and of any cardinality).*

Examples

Example

- $\mathbb{W} = C^* a^2 C^\omega \cup \mathrm{Inf}(a)$.



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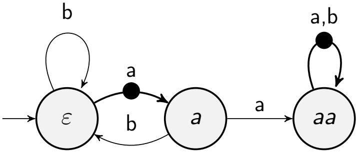

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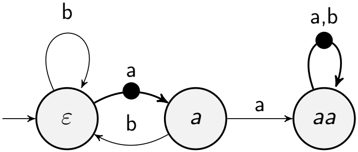

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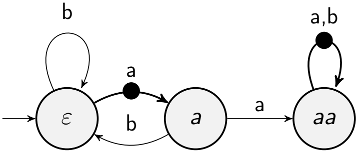

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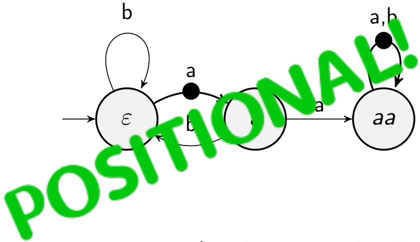

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- $\prec$ is total.
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Example

Remark: $\mathbb{W} = \text{Büchi}(a) \cup C^* a^2 C^\omega$ is not bi-positional:

$$C^\omega \setminus \mathbb{W} = (b^* ab)^* b^\omega \text{ (Not progress-consistent).}$$

Open questions

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Characterization of positionality for ω-regular languages.

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Subquestions:

- Characterization of positionality for languages recognized by deterministic co-Büchi automata.
- Union prefix-independent positional ω-regular conditions is positional?
- For ω-regular conditions, positionality over finite arenas implies positionality over arbitrary arenas?
- 1-to-2 players lift for ω-regular conditions.

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