

Characterising Memory in Infinite Games

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Abstract

This paper is concerned with games of infinite duration played over potentially infinite graphs. Recently, Ohlmann (TheoretCS 2023) presented a characterisation of objectives admitting optimal positional strategies, by means of universal graphs: an objective is positional if and only if it admits well-ordered monotone universal graphs. We extend Ohlmann’s characterisation to encompass (finite or infinite) memory upper bounds.

We prove that objectives admitting optimal strategies with ε -memory less than m (a memory that cannot be updated when reading an ε -edge) are exactly those which admit well-founded monotone universal graphs whose antichains have size bounded by m . We also give a characterisation of chromatic memory by means of appropriate universal structures. Our results apply to finite as well as infinite memory bounds (for instance, to objectives with finite but unbounded memory, or with countable memory strategies).

We illustrate the applicability of our framework by carrying out a few case studies, we provide examples witnessing limitations of our approach, and we discuss general closure properties which follow from our results.

2012 ACM Subject Classification Theory of computation \rightarrow Verification by model checking

Keywords and phrases Infinite duration games, Memory, Universal graphs

Digital Object Identifier 10.4230/LIPIcs.ICALP.2023.122

Category Track B: Automata, Logic, Semantics, and Theory of Programming

Related Version *Full Version*: <https://arxiv.org/abs/2209.12044>

Funding *Pierre Ohlmann*: European Research Council (ERC), grant agreement No 948057 – BOBR.



Acknowledgements We thank Nathanaël Fijalkow, Rémi Morvan and Pierre Vandenhove for stimulating discussions around the topic.

This document contains hyperlinks. Each occurrence of a notion is linked to its definition. On an electronic device, the reader can click on words or symbols (or just hover over them on some PDF readers) to see their definition.

1 Introduction

1.1 Context

Games and strategy complexity. We study zero-sum turn-based games on graphs, in which two players, that we call Eve and Adam, take turns in moving a token along the edges of a given (potentially infinite) edge-coloured directed graph. Vertices of the graph are partitioned into those belonging to Eve and those belonging to Adam. When the token lands in a vertex owned by player X, it is this player who chooses where to move next. This interaction, which is sometimes called a play, goes on in a non-terminating mode, producing an infinite sequence



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50th International Colloquium on Automata, Languages, and Programming (ICALP 2023).

Editors: Kousha Etessami, Uriel Feige, and Gabriele Puppis; Article No. 122; pp. 122:1–122:18

Leibniz International Proceedings in Informatics

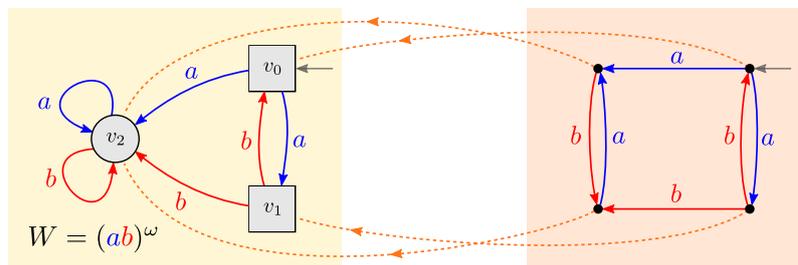


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of colours. We fix in advance an objective W , which is a language of infinite sequences of colours; plays producing a sequence of colours in W are considered to be winning for Eve, and plays that do not satisfy the objective W are winning for the opponent Adam.

In order to achieve their goal, players use strategies, which are representations of the course of all possible plays together with instructions on how to act in each scenario. In this work, we are interested in optimal strategies for Eve, that is, strategies that guarantee a victory whenever this is possible. More precisely, we are interested in the complexity of such strategies, or in other words, in the succinctness of the representation of the space of plays. The simplest strategies are those that assign in advance an outgoing edge to each vertex owned by Eve, and always play along this edge, disregarding all the other features of the play. All the information required to implement such a strategy appears in the game graph itself. These strategies are called positional (or memoryless). However, in some scenarios, playing optimally requires distinguishing different plays that end in the same vertex; one should remember other features of plays. An example of such a game is given in Figure 1.



■ **Figure 1** On the left, a game with objective $W = (ab)^\omega$; in words, Eve should ensure that the play alternates between a -edges and b -edges. We represent Eve’s vertices as circles and Adam’s as squares. On the right, a winning strategy for Eve which uses one state of memory for v_0 , one state of memory for v_1 , and two states of memory for v_2 . Note that two states of memory for v_2 are required here: a positional strategy would always follow the same self-loop and therefore cannot win. One can prove that any game with objective W which is won by Eve can be won even when restricting to strategies with two states of memory, that is, the memory requirements for W is exactly two.

Given an objective W , the question we are interested in is:

“What is the minimal strategy complexity required for Eve to play optimally in all games with objective W ?”

Positional objectives and universal graphs. As mentioned above, an important special case is that of positional objectives, those for which Eve does not require any memory to play optimally. A considerable body of research, with both theoretical and practical reach, has been devoted to the study of positionality. By now it is quite well-understood which objectives are positional for both players (bi-positional), thanks to the works of Gimbert and Zielonka [13] for finite game graphs, and of Colcombet and Niwiński [9] for arbitrary game graphs. However, a precise understanding of which objectives are positional for Eve – regardless of the opponent – remains somewhat elusive, even though this is a more relevant question in most application scenarios.

A recent progress in this direction was achieved by Ohlmann [19, 20], using totally ordered monotone universal graphs. Informally, an edge-coloured graph is universal with respect to a given objective W if it satisfies W (all paths satisfy W), and homomorphically embeds all graphs satisfying W . An ordered graph is monotone if its edge relations are monotone:

$$v \geq u \xrightarrow{c} u' \geq v' \implies v \xrightarrow{c} v', \text{ for every colour } c.$$

Ohlmann’s main result is a characterisation of positionality (assuming existence of a neutral letter): an objective is positional if and only if it admits well-ordered monotone universal graphs.

From positionality to finite memory. Positional objectives have good theoretical properties and do often arise in applications (in particular, parity, Rabin or energy objectives). It is also true, however, that this class lacks in expressivity and robustness: only a handful of objectives are positional, and very few closure properties are known to hold for positional objectives.

In contrast, objectives admitting optimal finite memory strategies are much more general; for instance they encompass all ω -regular objectives [14] (in fact, it was recently established [3] that optimal finite chromatic memory for both players characterises ω -regularity). Moreover, in practice, finite memory strategies can be implemented by means of a program, and memory bounds for Eve directly translates in space and time required to implement controllers, which gives additional motivation for their systematic study.

Formally, when moving from positionality to finite memory, a few modelling difficulties arise, giving rise to a few different notions. Most prominently, one may or may not include uncoloured edges (ε -edges) in the game, over which the memory state cannot be updated; additionally one may or may not restrict to chromatic memories, meaning those that record only the colours that have appeared so far. We now discuss some implications of these two choices.

It is known that allowing ε -edges impacts the difficulty of the games, in the sense that it may increase the memory required for winning strategies [5, 15, 23], thus leading to two different notions of memory (that we call ε -memory and ε -free memory). It is natural to wonder whether one of the two notions should be preferred over the other. We argue that allowing ε -edges turns out to be more natural in many applications. First, we notice that currently existing characterisations of the memory (for Muller objectives [12] and for topologically closed objectives [8]) do only apply to the case of ε -memory. More importantly, games induced by logical formulas in which players are interpreted as the existential player (controlling existential quantifiers and disjunctions) and the universal player (controlling universal quantifiers and conjunctions) naturally contain ε -edges (along which the memory indeed should not be allowed to be updated).

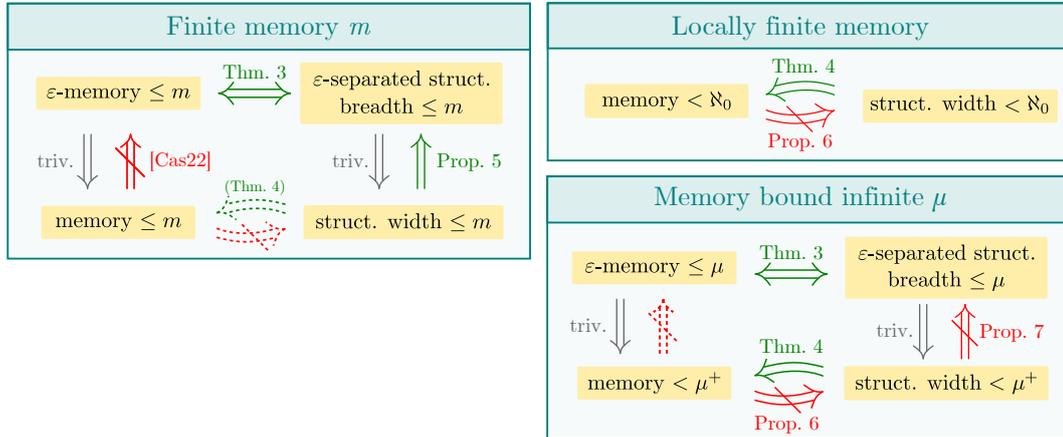
It was originally conjectured by Kopczyński [15] that chromatic strategies have the same power than non-chromatic ones. It was not until recently that this conjecture was refuted [5], and since then several works have provided new examples separating both notions [6, 17, 18]. It now appears from recent dedicated works [2, 3, 4, 5] that chromatic memory is an interesting notion in itself.

The main challenge in the study of strategy complexity is to prove upper bounds on memory requirements of a given objective. A great feature of Ohlmann’s result [20] is that it turns a question about games to a question about graphs, which are easier to handle. Despite its recent introduction, Ohlmann’s framework has already proved instrumental for deriving general positionality results in the context of objectives recognised by finite Büchi automata [1].

1.2 Contribution

The present paper builds on the aforementioned work of Ohlmann by extending it to encompass the more general setting of finite (or infinite) memory bounds. This yields the first known characterisation results for objectives with given memory bounds, and provides a (provably) general tool for establishing memory upper bounds.

Doing so requires relaxing from totally to partially ordered graphs, while keeping the same monotonicity requirement, along with some necessary technical adjustments. We essentially prove that the memory of an objective corresponds to the size of antichains in its well-founded monotone universal graph; however it turns out that the precise situation is more intricate. It is summed up in Figure 2 and explained in more details below.



■ **Figure 2** A summary of our main contributions. The three larger boxes correspond to the three regimes encompassed by our results: finite memory, locally finite memory and larger cardinal bounds. Each of the smaller boxes correspond to classes of objectives, where “struct.” stands for “existence of well-founded monotone universal graphs”; for example, the box labelled “ ε -separated struct. breadth $\leq m$ ” stands for “existence of ε -separated well-founded monotone universal graphs of breadth $\leq m$ ”. The dotted implications follow from combining other implications in the figure. For $m = 1$, all notions collapse to a single equivalence, which corresponds to Ohlmann’s characterisation.

It is convenient for us to define strategies directly as graphs (see Figure 1 for an example, and Section 2 for formal details), which allows us in particular to introduce new classes of objectives such as those admitting locally finite memory, discussed in more details below. For the well-studied case of finite memory bounds, our definition of memory coincides with the usual one.

Universal structures for memory. Our main contribution lies in introducing generalisations of Ohlmann’s structures, and proving general connections between existence of such universal structures for a given objective W , and memory bounds for W (Section 3).

The first variant we propose is obtained by relaxing the monotonicity requirement to partially ordered graphs; Theorem 4 states that (potentially infinite) bounds on antichains of a well-founded monotone universal graph translate to memory bounds.

The second variant we propose, called ε -separated structures, is tailored to capture ε -memory. These are monotone graphs where the partial order coincides with $\xrightarrow{\varepsilon}$ and is constrained to be a disjoint union of well-orders; the breadth of such a graph refers to the number of such well-orders. Theorem 3 states that the existence of such universal structures of breadth μ actually characterises having ε -memory $\leq \mu$. Additionally, we define chromatic ε -separated structures (over which each colour acts uniformly), and establish that they capture ε -chromatic memory.

Applying (infinite) Dilworth’s theorem we obtain that for finite m , one may turn any monotone graph of width m to an ε -separated one with breadth m (Proposition 5), and therefore in the setting of finite memory, the two notions collapse.

We are able to establish most (but not all) of our results in the more general framework of quantitative valuations; similarly as Ohlmann [20], we show how the notions instantiate in the qualitative case, and how they can be simplified assuming prefix-invariance properties.

Counterexamples for a complete picture. We provide additional negative results which set the limits of our approach, completing the picture in Figure 2. Namely, we build two families of counterexamples that are robust to larger cardinals; these give general separations of ε -free memory and ε -memory (Proposition 7), and negate the possibility of a converse for Theorem 4 (Proposition 6). This supports our informal claim that ε -memory is better behaved than ε -free memory.

Closure properties. Finally, we discuss how our characterisations can be exploited for deriving closure properties on some classes of objectives (Section 4). Apart from Ohlmann's result on lexicographic products of prefix-independent positional objectives [20], no such closure properties are known. Extending Ohlmann's proof to our framework, we prove that if W_1 and W_2 are prefix-independent objectives with ε -memory m_1 and m_2 , then their lexicographical product $W_1 \times W_2$ has ε -memory $\leq m_1 m_2$.

We then propose a new class of objectives with good properties, namely, objectives with locally finite memory: for each game, there exists a strategy which uses a finite (though possibly unbounded, even when the game is fixed) amount of memory states for each vertex. These objectives are connected with the theory of well-quasi orders (wqo), since they correspond to monotone universal graphs which are well-founded and have finite antichains. We obtain from the fact that wqo's are closed under intersections, that intersections of objectives with finite ε -memory have locally finite memory; an example is given by conjunctions of energy objectives which have unbounded finite memory even though energy objectives are positional. This hints at a general result, which is not implied by our characterisations but we conjecture to be true, that objectives with finite (possibly unbounded) memory are closed under intersection.

We end our paper by providing yet another application of our characterisation, establishing that prefix-independent Σ_2^0 objectives with finite memory are closed under countable unions. As of today, this is the only known (non-obvious) closure property pertaining to objectives with finite memory.

2 Preliminaries

For a finite or infinite word $w \in C^* \cup C^\omega$ we denote by w_i the letter at position i and by $|w|$ its length.

2.1 Graphs and morphisms

Graphs, paths and trees. A C -pregraph G , where C is a (potentially infinite) set of colours, is given by a set of vertices $V(G)$, and a set of coloured directed edges $E(G) \subseteq V(G) \times C \times V(G)$. We write $v \xrightarrow{c} v'$ for an edge (v, c, v') , say that it is outgoing from v , incoming in v' and has colour c . A C -graph G is a C -pregraph without sinks: from all $v \in V(G)$ there exists an outgoing edge $v \xrightarrow{c} v' \in E(G)$. We often say c -edges to refer to edges with colour c , and sometimes C' -edges for $C' \subseteq C$ for edges with colour in C' .

A *path* in a pregraph G is a finite or infinite sequence of edges of the form $\pi = (v_0 \xrightarrow{c_0} v_1)(v_1 \xrightarrow{c_1} v_2) \dots$, which for convenience we denote by $\pi = v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots$. We say that π is a path from v_0 in G . By convention, the empty path is a path from v_0 , for any $v_0 \in V(G)$. If π is a finite path, it is of the form $v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots \xrightarrow{c_{n-1}} v_n$, and in this case we say that it is a path from v_0 to v_n in G .

Given a subset $X \subseteq V(G)$ of vertices of a pregraph G , we let $G|_X$ denote the *restriction* of G to X , which is the graph given by $V(G|_X) = X$ and $E(G|_X) = E(G) \cap (X \times C \times X)$. Given a vertex $v \in V(G)$, we let $G[v]$ denote the restriction of G to vertices reachable from v .

A *C-tree* (resp. *C-pretree*) T is a C -graph (resp. C -pregraph) with an identified vertex $t_0 \in V(T)$ called its *root*, with the property that for each $t \in V(T)$, there is a unique path from t_0 to t . Note that since graphs have no sinks, trees are necessarily infinite. We remark that $T[t]$ represents the *subtree rooted at t* (if T is a tree, $T[t]$ is also a tree with root t).

When it is clear from context, we omit C and simply say “a graph” or “a tree”.

The *size* of a graph G (and by extension, of a tree) is the cardinality of $V(G)$.

Morphisms. A *morphism* ϕ between two graphs G and H is a map $\phi: V(G) \rightarrow V(H)$ such that for each edge $v \xrightarrow{c} v' \in E(G)$ it holds that $\phi(v) \xrightarrow{c} \phi(v') \in E(H)$. We write $\phi: G \rightarrow H$ in this case, and sometimes say that H embeds G . Note that morphisms preserve paths: if $v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots$ is a path in G , then $\phi(v_0) \xrightarrow{c_0} \phi(v_1) \xrightarrow{c_1} \dots$ is a path in H . An *isomorphism* is a bijective morphism whose inverse is a morphism; two graphs are isomorphic if they are connected by an isomorphism (stated differently, they are the same up to renaming the vertices). The composition of two morphisms is a morphism.

2.2 Valuations, games, strategies and memory

Valuations and objectives. A *C-valuation* is a map $\text{val}: C^\omega \rightarrow X$, where X is a complete linear order (that is, a total order in which all subsets have both a supremum and an infimum). The *value* $\text{val}_G(v_0)$ of a vertex $v_0 \in V(G)$ in a graph G is the supremum value of infinite paths from v_0 , where the value of an infinite path $\pi = v_0 \xrightarrow{c_0} v_1 \xrightarrow{c_1} \dots$ is defined to be $\text{val}(\pi) = \text{val}(c_0 c_1 \dots)$.

In the important special case where $X = \{\perp, \top\}$, $\perp < \top$, we identify val with $W = \text{val}^{-1}(\perp) \subseteq C^\omega$, and say that val (or W) is an *objective*. In a graph G , a path with value \perp (equivalently, whose sequence of colours belongs to W) is said to *satisfy W* , and a vertex v_0 with value \perp (equivalently, all paths from v_0 satisfy W) is also said to satisfy W . A graph is said to satisfy W if all its vertices satisfy it.

Games. A *C-game* is a tuple $\mathcal{G} = (G, V_{\text{Eve}}, v_0, \text{val})$, where G is a C -graph, V_{Eve} is a subset of $V(G)$, $v_0 \in V(G)$ is an identified initial vertex, and $\text{val}: C^\omega \rightarrow X$ is a C -valuation. We interpret V_{Eve} to be the set of vertices controlled by the first player, *Eve*, and we will write $V_{\text{Adam}} = V(G) \setminus V_{\text{Eve}}$ for the vertices controlled by her opponent, *Adam*. A game is played as follows: starting from v_0 , successive moves are played where the player controlling the current vertex v chooses an outgoing edge $v \xrightarrow{c} v'$ and proceed to v' . This interaction goes on forever, producing an infinite path π from v_0 . Eve’s goal is to minimise the value of the produced path π , whereas Adam aims to maximise it.

In this paper, we are interested in questions of strategy complexity for Eve: if she wins, how much memory is required/sufficient? Formally, these are independent of questions of determinacy (is there a winner?). As a result, we will only ever consider strategies for Eve.

Strategies. A *strategy* in the game \mathcal{G} is a tuple $\mathcal{S} = (S, \pi_{\mathcal{S}}, s_0)$ where S is a graph, $\pi_{\mathcal{S}}$ is a morphism $\pi_{\mathcal{S}}: S \rightarrow G$ called the \mathcal{S} -*projection* and $s_0 \in V(S)$ satisfying:

- $\pi_{\mathcal{S}}(s_0) = v_0$,
- for all $v \in V_{\text{Adam}}$, all outgoing edges $v \xrightarrow{c} v' \in E(G)$ and all $s \in \pi_{\mathcal{S}}^{-1}(v)$, there is $s' \in \pi_{\mathcal{S}}^{-1}(v')$ such that $s \xrightarrow{c} s' \in E(S)$.

Note that the requirements that S is a graph and $\pi_{\mathcal{S}}$ a morphism impose that for all $v \in V_{\text{Eve}}$ and $s \in \pi_{\mathcal{S}}^{-1}(v)$, s has an outgoing edge $s \xrightarrow{c} s' \in E(S)$ satisfying $\pi_{\mathcal{S}}(s) = v \xrightarrow{c} \pi_{\mathcal{S}}(s') \in E(G)$.

We remark that we do not impose that for each $v \in V_{\text{Eve}}$ and $s \in \pi_{\mathcal{S}}^{-1}(v)$, s has *exactly one* outgoing edge. Stated differently, non-determinism is allowed in this definition of strategy. As the upcoming definition of value of a strategy will clarify, we can interpret that Adam decides how to resolve this non-determinism.

On an informal level, a strategy $\mathcal{S} = (S, \pi_{\mathcal{S}}, s_0)$ from $v_0 \in G$ is used by Eve to play in the game \mathcal{G} as follows:

- whenever the game is in a position $v \in V(G)$, the strategy is in a position $s \in \pi_{\mathcal{S}}^{-1}(v)$;
- initially, the position in the game is v_0 , and the position in the strategy is $s_0 \in \pi_{\mathcal{S}}^{-1}(v_0)$;
- if the position v in the game belongs to V_{Adam} , and Adam chooses the edge $v \xrightarrow{c} v'$ in G , then the strategy state is updated following an edge $s \xrightarrow{c} s'$ in S with $\pi_{\mathcal{S}}(s') = v'$, which exists by definition of \mathcal{S} (if multiple options exist, Adam chooses one);
- if the position v in the game belong to V_{Eve} , then the strategy specifies at least one successor $s \xrightarrow{c} s'$ from the current $s \in \pi_{\mathcal{S}}^{-1}(v)$, and the game proceeds along the edge $v \xrightarrow{c} \pi_{\mathcal{S}}(s')$ (if multiple options exist in the strategy, which corresponds to the non-determinism mentioned above, then Adam chooses one).

Note that infinite sequences of colours produced when playing as above are exactly labels of infinite paths from s_0 in S .

The *value* $\text{val}(\mathcal{S})$ of a strategy \mathcal{S} is $\text{val}_{\mathcal{S}}(s_0)$. The *value* $\text{val}(\mathcal{G})$ of a game is the infimum value among its strategies. If val is an objective, we say that \mathcal{S} is *winning* if $\text{val}_{\mathcal{S}}(s_0) = \perp$, and we say that Eve *wins* a game \mathcal{G} if $\text{val}(\mathcal{G}) = \perp$.

The following observation is standard (it is usually taken as the definition of a strategy).

► **Lemma 1.** *The value of a game is reached with strategies that are trees.*

Memory. For a strategy $\mathcal{S} = (S, \pi_{\mathcal{S}}, s_0)$, we interpret the fibres $\pi_{\mathcal{S}}^{-1}(v)$ as memory spaces. Given a cardinal μ , we say that \mathcal{S} has *memory strictly less than μ* , (resp. *less than μ*) if for all $v \in V(G)$, $|\pi_{\mathcal{S}}^{-1}(v)| < \mu$ (resp. $|\pi_{\mathcal{S}}^{-1}(v)| \leq \mu$). As it will appear later on, it is convenient for us to be able to use both strict and non-strict inequalities. By means of clarity and conciseness, we usually simply write “ \mathcal{S} has memory $< \mu$ ” (resp. $\leq \mu$) instead of “ \mathcal{S} has memory strictly less than μ (resp. less than μ)”.

We say that a valuation val has *memory strictly less than μ* , or $< \mu$, (resp. *less than μ* , or $\leq \mu$) if in all games with valuation val , the value is reached with strategies with memory $< \mu$.

Conversely, we say that val has *memory at least μ* , or $\geq \mu$, if it does not have memory $< \mu$: there exists a game with valuation val in which Eve cannot reach the value with strategies with memory $< \mu$.

We say that val is *positional* if it has memory ≤ 1 .

Product strategies, chromatic strategies. A strategy $\mathcal{S} = (S, \pi_{\mathcal{S}}, s_0)$ in the game \mathcal{G} is a *product strategy* over a set M if $V(S) \subseteq V(G) \times M$, with $\pi_{\mathcal{S}}(v, m) = v$. We call the elements of M *memory states*. Note that the memory in a product strategy over M is $\leq |M|$, since

fibers are included in M . A product strategy is *chromatic* if there is a map $\delta : M \times C \rightarrow M$ such that for all $(v, m) \xrightarrow{c} (v', m') \in E(S)$ we have $m' = \delta(m, c)$. We say in this case that δ is the *update function* of \mathcal{S} . In words, the update of the memory state in a chromatic strategy depends only on the current memory state and the colour that is read. A valuation val has *chromatic memory* $< \mu$ (resp. $\leq \mu$) if in all games with valuation val , the value is reached with chromatic strategies with memory $< \mu$ (resp. $\leq \mu$).

ε -games and ε -strategies. Fix a set of colours C , a fresh colour $\varepsilon \notin C$, and let $C^\varepsilon = C \sqcup \{\varepsilon\}$. The C -*projection* of an infinite sequence $w \in (C^\varepsilon)^\omega$ is the (finite or infinite) sequence $w_C \in C^* \cup C^\omega$ obtained by removing all ε 's in w . Given a C -valuation $\text{val} : C^\omega \rightarrow X$, define its ε -*extension* val^ε to be given by

$$\text{val}^\varepsilon(w) = \begin{cases} \text{val}(w_C), & \text{if } |w_C| = \infty, \\ \inf_{w' \in C^\omega} \text{val}(w_C w'), & \text{otherwise.} \end{cases}$$

It is the unique extension of val with ε as a strongly neutral colour, in the sense of Ohlmann [20]. In particular, if W is an objective and $w \in C^*$, $w\varepsilon^\omega \in W^\varepsilon$ unless w has no winning continuation in W .

An ε -*game* \mathcal{G} is a C^ε -game with valuation val^ε . An ε -*strategy* over such a game is a product strategy $\mathcal{S} = (S, \pi_S, s_0)$ over some set M such that $(v, m) \xrightarrow{\varepsilon} (v', m') \in E(S)$ implies $m = m'$. Intuitively, Eve is not allowed to update the state of the memory when an ε -edge is traversed. The *memory of an ε -strategy* is defined to be $|M|$. A valuation val has ε -*memory* $< \mu$ (resp. $\leq \mu$) if in all ε -games with valuation val^ε , the value is attained by ε -strategies with memory $< \mu$ (resp. $\leq \mu$).

Note that by definition, a chromatic strategy over M with update function δ is an ε -strategy if and only if for all $m \in M$ it holds that $\delta(m, \varepsilon) = m$. We call such a strategy an ε -*chromatic strategy*. A valuation val has ε -*chromatic memory* $< \mu$ (resp. $\leq \mu$) if in all ε -games with valuation val^ε , the value is attained by ε -chromatic strategies with memory $< \mu$ (resp. $\leq \mu$).

Whenever we want to emphasise that we consider games (resp. strategies, memory) without ε , we might add the adjective ε -*free*.

2.3 Monotonicity and universality

Monotonicity. A partially ordered graph (G, \leq) is *monotone* if

$$u \geq v \xrightarrow{c} v' \geq u' \text{ implies } u \xrightarrow{c} u' \text{ in } G.$$

A partially ordered graph (G, \leq) is called *well-monotone* if it is monotone and it is well-founded as a partial order. We say that the *width* of a partially ordered graph is $< \mu$ (resp. $\leq \mu$) if it does not contain antichains of size μ (resp. of size strictly greater than μ).

ε -separation. An ε -*separated monotone graph* over a set M is a C^ε -graph G such that $\xrightarrow{\varepsilon}$ defines a partial order making G monotone ($v \leq v' \iff v' \xrightarrow{\varepsilon} v \in E(G)$), and moreover $V(G)$ is partitioned into $(V_m)_{m \in M}$ such that for all $m \in M$, $\xrightarrow{\varepsilon}$ induces a total order over V_m , and there are no ε -edges between different parts: $v \xrightarrow{\varepsilon} v' \in E(G)$ implies that $v, v' \in V_m$ for some $m \in M$. See Figure 3. We define the *breadth* of such a graph as $|M|$.

An ε -separated monotone graph G over M is *chromatic* if there is a map $\delta : M \times C \rightarrow M$ such that for all $v \xrightarrow{c} v' \in E(G)$ with $v \in V_m$ and $v' \in V_{m'}$ we have $m' = \delta(m, c)$. We also say in this case that δ is the *update function* of G .

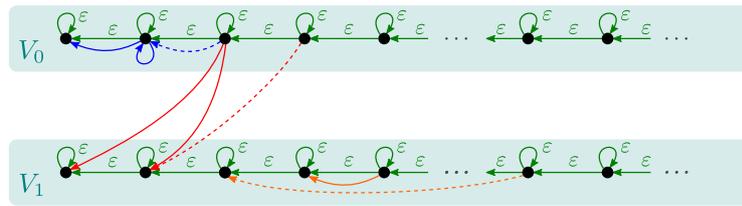


Figure 3 An ε -separated chromatic monotone graph of breadth 2. Note that $\xrightarrow{\varepsilon}$ defines a total order on each V_i (edges following from transitivity are not represented). Many edges which follow from monotonicity are not depicted, the dotted edges give a few examples.

Universality. Given a C -valuation val , a C -graph G and a cardinal κ , we say that G is (κ, val) -universal if for all C -trees T of cardinality $< \kappa$, there exists a morphism $\phi : T \rightarrow G$ such that $\text{val}_G(\phi(t_0)) \leq \text{val}_T(t_0)$, where t_0 is the root of T . We say that ϕ *preserves the value* at the root to refer to this property (we remark that, in that case, $\text{val}_G(\phi(t_0)) = \text{val}_T(t_0)$, since the other inequality always holds).

► **Remark 2.** An example where the definition with graphs is too constrained to capture memory is given in Proposition 22 from the full version [7].

3 Universal structures characterise memory

Statement of the main results. We start with our characterisations of ε -memory and ε -chromatic memory via (chromatic) ε -separated universal graphs.

► **Theorem 3.** *Let val be a valuation. If for all cardinals κ there exists an ε -separated (chromatic) and well-monotone $(\kappa, \text{val}^\varepsilon)$ -universal graph of breadth $\leq \mu$, then val has ε -chromatic-memory $\leq \mu$. The converse holds if val is an objective (in both the chromatic and non-chromatic cases).*

Our second result concerns ε -free memory. It is stated with strict inequalities, which are relevant in this case and allow for more precision. However, we do not have a converse statement; in fact, the converse cannot hold (see also Figure 2 and Proposition 7).

► **Theorem 4.** *Let val be a valuation. If for all cardinals κ there exists a well-monotone (κ, val) -universal graph of width $< \mu$, then val has ε -free memory $< \mu$.*

As we will see in Proposition 5, the two results above collapse for finite cardinals.

We give the main ideas of the proofs of these two theorems, in both cases we extend the proofs from Ohlmann [20]. The full proofs can be found in Sections 3.2 and 3.3 in the full version [7].

Proof sketch of Theorem 4 and of \implies in Theorem 3. We discuss the proof of Theorem 4 (the proof of the first implication in Theorem 3 follows the same structure). In this case, assuming existence of a universal structure, we prove upper bounds in the memory of a valuation. This is done using a strategy-folding procedure that is guided by the morphism towards the universal structure. Let (U, \leq) be a well-monotone (κ, val) -universal graph of width $< \mu$. Suppose that \mathcal{G} is a game of cardinality $\leq \kappa$ with valuation val , and let $\mathcal{T} = (T, \pi_{\mathcal{T}}, t_0)$ be a strategy for Eve given by a tree. By universality of U , there is a morphism $\phi : T \rightarrow U$ preserving the value at the root of T .

122:10 Characterising Memory in Infinite Games

For each vertex v of the game we consider the set $\phi(\pi_{\mathcal{T}}^{-1}(v))$ in U . Since U is well-founded and of width $< \mu$, the set M_v of minimal elements of $\phi(\pi_{\mathcal{T}}^{-1}(v))$ has size strictly less than μ . This allows us to define a strategy over $\bigcup_v \{v\} \times M_v$ which simulates the strategy \mathcal{T} as follows: we take a representative $t_{(v,m)} \in \pi_{\mathcal{T}}^{-1}(v)$, and for each $m \in M_v$, we follow the decisions made at $t_{(v,m)}$ when we are in (v, m) .

To define the update of the memory, for each move $v \xrightarrow{c} v' \in E(G)$ and edge $t_{(v,m)} \xrightarrow{c} t' \in E(T)$, we consider the image $\phi(t') \in U$. By definition, there is an element m' in $M_{v'}$ smaller than $\phi(t')$, so we let $(v, m) \xrightarrow{c} (v', m')$. By monotonicity it follows that this strategy has the same value than \mathcal{T} . If U is assumed to be (chromatic) ε -separated, it follows directly that the obtained strategy is a (chromatic) ε -strategy. \blacktriangleleft

Proof sketch of \Leftarrow in Theorem 3. We prove the following result: given a C^ε -tree T satisfying an objective W , there exists an ε -separated well-monotone graph U of breadth $\leq \mu$ and a morphism $T \rightarrow U$ preserving the value at the root. Once this is proved, applying it to the tree $T_{U^{\text{niv}}}$ consisting of a root connected by an ε -edge to every C^ε -tree $< \kappa$ satisfying W yields a (κ, W^ε) -universal graph.

In order to prove this result, we consider the following game: Adam controls the vertices from T , and for each non-empty set $A \subseteq V(T)$, we add a vertex v_A controlled by Eve with ε -edges back and forth from any vertex in A . This game is won by Eve: whenever Adam chooses an edge $t \xrightarrow{\varepsilon} v_A$ she just need to respond $v_A \xrightarrow{\varepsilon} t$. Since W has ε -memory $\leq \mu$, Eve has a winning ε -strategy \mathcal{S} over $V(S) = V(G) \times M$ with $|M| \leq \mu$.

We define the wanted morphism $\phi: T \rightarrow S$ in a top-down fashion using the properties of a strategy: $\phi(t_0) = s_0$ and if $\phi(t) = (t, m)$ and $t \xrightarrow{c} t' \in E(T)$, we set $\phi(t') = (t', m')$ where m' is such that $(t, m) \xrightarrow{c} (t', m') \in E(S)$. This morphism preserves the value of t_0 , because \mathcal{S} is a winning strategy. With some addition technical tweaks we transform S into an ε -separated graph U of breadth $\leq \mu$ while maintaining a value-preserving morphism $\phi: T \rightarrow U$. \blacktriangleleft

Applying Dilworth's Theorem [11], we prove that the two notions of graphs collapse (both characterise ε -memory) when dealing with objectives and finite memory bounds.

► Proposition 5. *Let W be an objective and $m \in \mathbb{N}$. If for all cardinals κ there exists a well-monotone graph which is (κ, W) -universal and has width $\leq m$, then for all cardinals κ there is also an ε -separated well-monotone (κ, W^ε) -universal graph of breadth $\leq m$, and therefore W has ε -memory $\leq m$.*

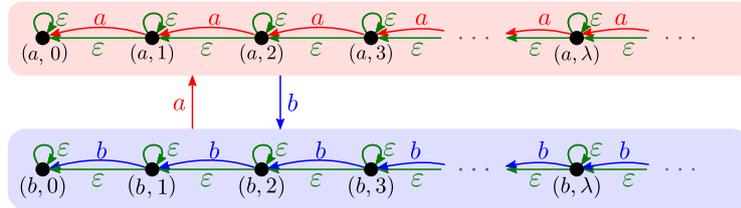
An objective $W \subseteq C^\omega$ is said to be *prefix-independent* if for all colours $c \in C$ it holds that $cW = W$. It is not difficult to prove (this was already done by Ohlmann [20]) that when considering prefix-independent objectives, one can use a simpler definition of universality, namely, a graph U is (κ, W) -universal (for prefix-independent objectives) if U satisfies W and embeds any tree of cardinality $< \kappa$ which satisfies W .

Some concrete examples. We start by illustrating the notions presented until now and some methods to derive universality proofs with a few simple concrete examples of objectives.

For many more examples, as well as the missing proofs of this paragraph, we refer to the Section 4 of the full version [7]. There, we also re-obtain in our framework the general characterisations of [8] for topologically closed objectives, and of [12] for Muller objectives.

Objective $W_1 = \{w \in \{a, b\}^\omega \mid a \text{ and } b \text{ occur infinitely often in } w\}$. We show, for each cardinal κ , an ε -separated chromatic and well-monotone $(\kappa, W_1^\varepsilon)$ -universal graph of breadth 2. This implies that the ε -chromatic memory of W_1 is at most 2.

Fix a cardinal number κ and consider the graph U from Figure 4. It is easy to check that U is an ε -separated monotone graph over the set $M = \{a, b\}$ and that it is indeed chromatic and satisfies W . We sketch a universality proof. Since W_1 is prefix-independent, we show that U embeds any tree of cardinality $< \kappa$ which satisfies W .



■ **Figure 4** Universal graph for W_1 . The order coincides with $\xrightarrow{\varepsilon}$ (as required by the definition of ε -separated graphs). Edges following from monotonicity are not represented. An edge between boxes indicates that all edges are put between vertices in the respective boxes.

Let T be a C -tree of size $< \kappa$ which satisfies W , and let t_0 be its root. Note that all paths from t_0 eventually visit a b -edge; there is in fact an ordinal $\lambda_0 < \kappa$ (defined by induction) which counts the maximal amount of a -edges seen from t_0 before a b -edge is seen; we set $\phi(t_0)$ to be (a, λ_0) .

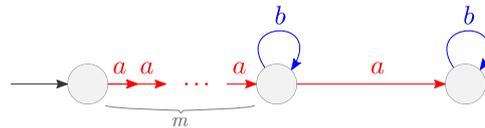
Then for each edge $t_0 \xrightarrow{c} t \in E(T)$ we proceed as follows.

- If $c \in \{a, \varepsilon\}$, we iterate exactly the same process on t , but the ordinal count will on the number of a 's will have decreased (or even strictly decreased if $c = a$) from t_0 to t , which guarantees that $\phi(t_0) \xrightarrow{a} \phi(t)$ is indeed an edge in U .
- If $c = b$, then we iterate the same process of t but inverting the roles of a and b ; thus $\phi(t)$ is of the form (b, λ_b) for some $\lambda_b < \kappa$, and the edge $\phi(t_0) \xrightarrow{b} \phi(t)$ belongs to U , as required.

This concludes the top-down construction of ϕ and the universality proof.

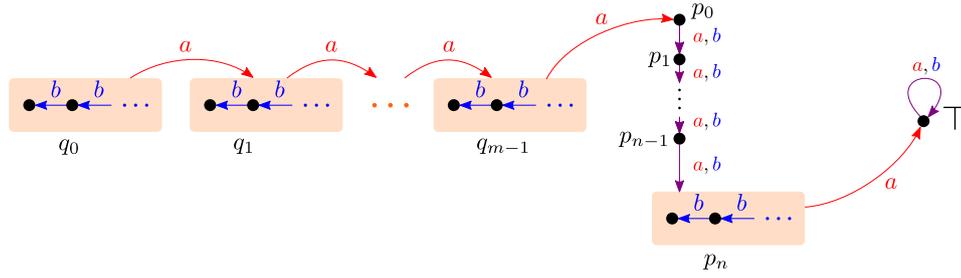
It is not difficult to find lower bounds to see that the ε -free memory of W_1 is ≥ 2 . For example, a game with just one vertex controlled by Eve where she can choose to produce a or b provides this lower bound. Therefore, the exact memory of W_1 is 2, for all the different notions of memory.

Objective $W_2 = (C^*a)^m C^{\geq n} a C^\omega$ with $C = \{a, b\}$ and $m, n \geq 1$. We provide a universal graph of width $n + 1$ which proves that the ε -memory is $\leq n + 1$. A matching lower bound on the ε -free memory follows from the game depicted on Figure 5. We remark that from the minimal automaton for the regular language $L = (C^*a)^m C^{\geq n} a$ we only obtain an upper bound of $n + m + 1$ on the memory.



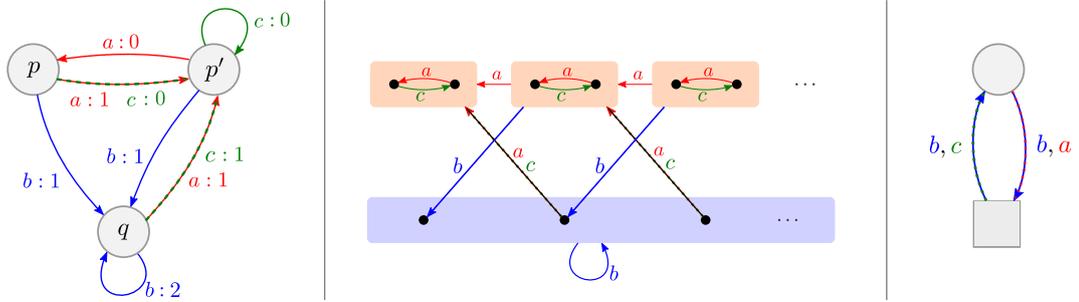
■ **Figure 5** A game where Eve requires memory $n + 1$ to ensure objective W_2 .

The well-monotone graph U depicted in Figure 6 proves the $n + 1$ upper bound on the ε -memory. Actually, it turns out that even the ε -chromatic memory of W_2 is $n + 1$, which requires a more subtle construction presented in the full version.



■ **Figure 6** A well-monotone graph U which has width $n + 1$ and is universal for W_2 .

Objective $W_3 = \{w \in C^\omega \mid w \text{ contains infinitely often } bb \text{ or (finitely often } b \text{ and } aa)\}$ over $C = \{a, b, c\}$. Figure 7 depicts a deterministic parity automaton \mathcal{A} of size 3 recognising W_3 ; this gives an upper bound of 3 on the memory of W_3 . The game depicted on the right of Figure 7 witnesses that Eve requires ε -free memory ≥ 2 : positional strategies are losing, but she wins by answering b to b and a to c .



■ **Figure 7** On the left, a deterministic parity automaton \mathcal{A} with three states recognising W_3 (we use max-parity semantics). In the middle, an ε -separated chromatic universal graph U of breadth 2 for W_3 ; as always, edges following from monotonicity are omitted. On the right, a game witnessing that Eve requires ε -free memory ≥ 2 .

The graph U depicted in the middle of Figure 7 is an ε -separated chromatic well-monotone universal graph for W_3 of breadth 2, providing the upper bound of 2 on all the types of memory for W_3 .

Counterexamples. We now provide two negative results. First, we show that the converse of Theorem 4 does not hold, even in the case of objectives.

► **Proposition 6.** For each cardinal μ , the objective $W_\mu = \{w_0 w_1 \dots \in \mu^\omega \mid \forall i, w_i \neq w_{i+1}\}$ satisfies that

1. the ε -free memory of W_μ is ≤ 2 ;
2. the ε -free memory of W_μ^ε is $\geq \mu$; and therefore the ε -memory of W_μ is $\geq \mu$; and
3. there is κ such that any monotone (κ, W_μ) -universal graph has width $\geq \mu$.

Second, we prove that Proposition 5 cannot hold if the bound on the size of the antichains of the graph is not finite.

► **Proposition 7.** For any infinite cardinal μ , the objective $W_\mu = \{(w, w') \in (\mu \times \mu)^\omega \mid \nexists i \text{ such that } w_i < w_{i+1} \text{ and } w'_i < w'_{i+1}\}$ is such that

- for all cardinals κ there exists a well-monotone (κ, W_μ) -universal graph whose antichains have cardinality $< \aleph_0$; and
- there is an ε -game with objective W_μ^ε requiring ε -memory $\geq \mu$.

4 Closure properties

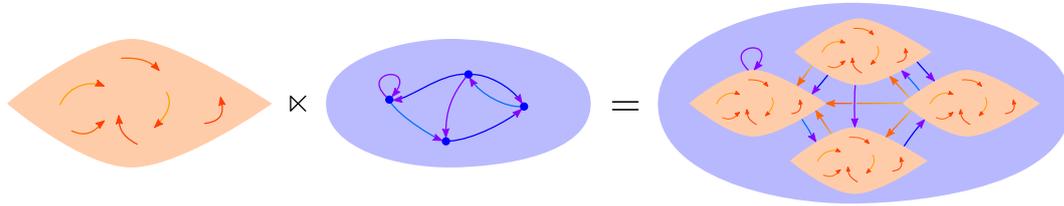
Lexicographical products. We provide a study of lexicographical products, as introduced by Ohlmann [20], whose result we generalize to finite memory bounds.

Given two prefix-independent objectives W_1 and W_2 over disjoint sets of colours C_1 and C_2 , we define their *lexicographical product* $W_1 \times W_2$ over $C = C_1 \sqcup C_2$ by

$$W_1 \times W_2 = \{w \in C^\omega \mid [w^2 \text{ is infinite and in } W_2] \text{ or } [w^2 \text{ is finite and } w^1 \in W_1]\},$$

where w^1 (resp. w^2) is the (finite or infinite) word obtained by restricting w to occurrences of letters from C_1 (resp. C_2) in the same order. Note that if w^2 is finite then w^1 is infinite, which is why the product is well defined.

We now define the *lexicographical product* (U, \leq) of two ordered graphs (U_1, \leq_1) and (U_2, \leq_2) . Intuitively, each vertex in U_2 is replaced by a copy of U_1 . (see also Figure 8).



■ **Figure 8** Illustration of the lexicographical product of two ordered graphs.

Formally $U_1 \times U_2 = U$ is defined over the lexicographical product of $(V(U_1), \leq_1)$ and $(V(U_2), \leq_2)$, that is $V(U) = V(U_1) \times V(U_2)$ and \leq is the lexicographical product of \leq_1 and \leq_2 . Its edges are:

$$E(U) = \{(u_1, u_2) \xrightarrow{c_1} (u'_1, u'_2) \mid c_1 \in C_1 \text{ and } (u_2 >_2 u'_2 \text{ or } [u_2 = u'_2 \text{ and } u_1 \xrightarrow{c_1} u'_1])\} \cup \{(u_1, u_2) \xrightarrow{c_2} (u'_1, u'_2) \mid c_2 \in C_2 \text{ and } u_2 \xrightarrow{c_2} u'_2\}.$$

We now state our main result in this section, a direct extension of [20, Theorem 18].

► **Theorem 8.** *Let W_1 and W_2 be two prefix-independent objectives over disjoint sets of colours C_1 and C_2 . Let κ be a cardinal and let (U_1, \leq) and (U_2, \leq) be monotone graphs which are respectively (κ, W_1) and (κ, W_2) -universal. Then $U_1 \times U_2$ is monotone and $(\kappa, W_1 \times W_2)$ -universal.*

Using Theorems 3 and 4 together with Proposition 5, we deduce the following result.

► **Corollary 9.** *Let W_1 and W_2 be two prefix-independent objectives over disjoint sets of colours C_1 and C_2 , and assume that W_1 (resp. W_2) has ε -memory $\leq n_1 \in \mathbb{N}$ (resp. $\leq n_2$). Then, their lexicographical product $W_1 \times W_2$ has ε -memory $\leq n_1 n_2$.*

Combining objectives with locally finite memory. When applied to $\mu = \aleph_0$, since well-founded orders with bounded antichains correspond to well-quasi-orders (wqo's), Theorem 4 states that the existence of universal monotone graphs which are wqo's for a given objective (or even, a valuation) entails *locally finite memory*, meaning that for any ε -free game there is an optimal strategy \mathcal{S} such that for all vertices v , the amount of memory used at v (that is, the cardinality of $\pi_{\mathcal{S}}^{-1}(v)$) is finite. Unfortunately this is not a characterisation: Proposition 6 applied to $\mu = \aleph_0$ gives an objective with ε -free memory 2 but which does not admit such universal structures. Still, by combining our knowledge so far with a few additional insights

122:14 Characterising Memory in Infinite Games

stated below, we may derive some strong closure properties pertaining to objectives with locally finite memory. In the sequel, we will simply say *monotone wqo* for a well-monotone graph whose antichains are finite.

Given two partially ordered sets (U_1, \leq_1) and (U_2, \leq_2) , we define their (*direct*) *product* to be the partially ordered set $(U_1 \times U_2, \leq)$, where

$$(u_1, u_2) \leq (u'_1, u'_2) \iff [u_1 \leq u'_1] \text{ and } [u_2 \leq u'_2].$$

Note that if \leq_1 and \leq_2 are well-founded, then so is \leq . However, there may be considerable blowup on the size of antichains, for instance, $\omega \times \omega$ has arbitrarily large (finite) antichains whereas ω is a total order. However, it is a well-known fact that the product of two wqo's is also a wqo (see for instance [10]), that is, one may not go from finite to infinite antichains.

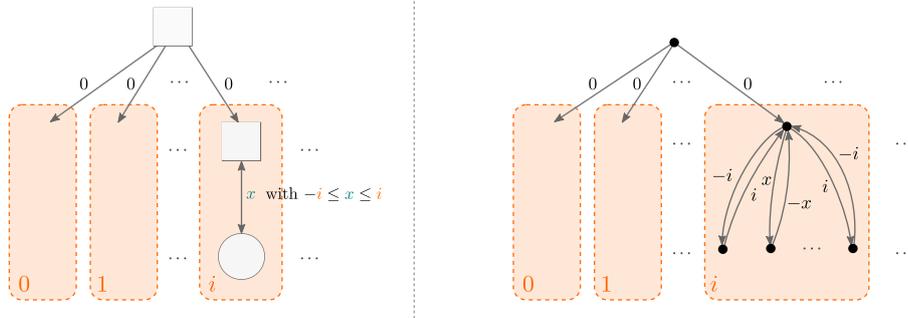
Given two partially ordered C -graphs (G_1, \leq_1) and (G_2, \leq_2) , we define their (*direct*) *product* to be the partially ordered C -graph G defined over the product of $(V(G_1), \leq_1)$ and $(V(G_2), \leq_2)$ by

$$E(G) = \{(v_1, v_2) \xrightarrow{c} (v'_1, v'_2) \mid v_1 \xrightarrow{c} v'_1 \in E(G_1) \text{ and } v_2 \xrightarrow{c} v'_2 \in E(G_2)\}.$$

Note that if (G_1, \leq_1) and (G_2, \leq_2) are monotone, then so is their product. Therefore, if (G_1, \leq_1) and (G_2, \leq_2) are monotone wqo's, then so is their product. Our discussion hinges on the following result.

► **Lemma 10.** *Let κ be a cardinal, and $W_1, W_2 \subseteq C^\omega$ be two objectives. Let (U_1, \leq_1) and (U_2, \leq_2) be two C -graphs which are (κ, W_1) and (κ, W_2) -universal, respectively. Then their product U is $(\kappa, W_1 \cap W_2)$ -universal.*

Therefore, by combining this lemma with the fact that wqo's are closed under product, we obtain that if two objectives W_1 and W_2 have monotone wqo's as universal graphs, then so does their intersection, hence, from Theorem 4, $W_1 \cap W_2$ has locally finite memory. In particular, thanks to Theorem 3, we get the following weak closure property.



■ **Figure 9** A game where initially, Adam chooses an upper bound i , then the players alternate in choosing integers in $[-i, i]$. Eve wins if the partial sums of the weights remain bounded both from above and below (bi-boundedness objective). She can ensure a win by simply playing the opposite of Adam in each round (this strategy is represented on the right-hand side), which requires unbounded but locally finite memory. Since bi-boundedness objectives are intersections of two positional objectives (being bounded from above and from below), our results in this section ensure that any game with a bi-boundedness objective has optimal locally finite memory strategies.

► **Corollary 11.** *Let W_1 and W_2 be two objectives which have monotone wqo's as universal graphs. Then so does $W_1 \cap W_2$. In particular the intersection of two objectives with finite ε -memory has locally finite memory.*

The upper bound stated in Corollary 11 is met: Figure 9 gives an example where W_1 and W_2 are positional but $W_1 \cap W_2$ has ε -free memory $> n$ for all $n \in \mathbb{N}$.

Although our results fall short of implying such a strong closure property, we may still state the following conjecture:

► **Conjecture 12.** *Objectives with ε -free memory $< \aleph_0$ are closed under intersection.*

Unions of prefix-independent Σ_2^0 objectives. The Cantor topology on C^ω naturally provides a way to define general families of objectives that have been well-studied in the literature of formal languages (we refer to [21] for a general overview). In particular, some of these classes of objectives are given by the different levels of the Borel hierarchy; the lowest levels are Σ_1^0 , consisting on the open subsets, and Π_1^0 , consisting on the closed subsets. The level Σ_{n+1}^0 (resp. Π_{n+1}^0) contains the countable unions (resp. countable intersections) of subsets in Π_n^0 (resp. Σ_n^0).

We now prove that prefix-independent objectives in Σ_2^0 with ε -memory $\leq m \in \mathbb{N}$ are closed under countable unions. We recall that Σ_2^0 objectives are those of the form $W_{\mathcal{L}} = \{w \in C^\omega \mid w \text{ has finitely many prefixes in } \mathcal{L}\}$, where $\mathcal{L} \subseteq C^*$ is an arbitrary language of finite words [22].

► **Theorem 13.** *Prefix-independent Σ_2^0 objectives with ε -memory $\leq m \in \mathbb{N}$ are closed under countable unions.*

Our proof relies on the definition of the *direct sum* of a family of universal graphs (obtained by concatenating them) and the following lemma.

► **Lemma 14.** *Let $W_0, W_1, \dots \subseteq C^\omega$ be prefix-independent Σ_2^0 objectives, κ be a cardinal, and U_0, U_1, \dots be C -graphs such that for each i , U_i is (W_i, κ) -universal. Let $W = \bigcup_i W_i$. Then the graph $U \times \kappa$, where U is the direct sum of the U_i 's, and κ is the edgeless graph with κ vertices, is (κ, W) -universal.*

Proof sketch. Let T be a tree of cardinality $< \kappa$ satisfying W . Since W is prefix-independent, proving that there is $t \in V(T)$ inducing a subtree $T[t]$ such that $T[t] \rightarrow U$ is enough to derive universality of $U \times \kappa$ (in the full version [7], this useful fact is stated as Lemma 10). Since U is the direct sum of the U_i 's and since each U_i is κ -universal for W_i , this amounts to showing that there is $i \in \mathbb{N}$ and $t \in T$ such that $T[t]$ satisfies W_i . Assume otherwise. Take $e = e_0 e_1 \dots \in \mathbb{N}^\omega$ to be a word over the naturals with infinitely many occurrences of each natural, for instance $e = 010120123\dots$. For each $i \in \mathbb{N}$, let $\mathcal{L}_i \subseteq C^*$ be such that $W_i = \{w \in C^\omega \mid w \text{ has finitely many prefixes in } \mathcal{L}_i\}$.

We now construct an infinite path $\pi = \pi_0 \pi_1 \dots$ starting from the root t_0 in T such that for each i , the coloration $w_0 \dots w_i$ of $\pi_0 \dots \pi_i$ belongs to \mathcal{L}_{e_i} . This implies that the coloration w of π has infinitely many prefixes in each of the \mathcal{L}_i 's, therefore it does not belong to W , a contradiction. Assume $\pi = \pi_0 \dots \pi_{i-1} : t_0 \xrightarrow{w_0 \dots w_{i-1}} t$ constructed up to π_{i-1} . Since by assumption, $T[t]$ does not satisfy W_{e_i} , there is a path $\pi' : t \xrightarrow{w}$ such that $w \notin W_{e_i}$. By prefix-independence of W_{e_i} , we get $w_0 \dots w_{i-1} w \notin W_{e_i}$, thus w has a prefix w_i such that $w_0 \dots w_{i-1} w_i \in \mathcal{L}_{e_i}$; this allows us to augment π as required and conclude our proof. ◀

The theorem follows from Lemma 14, Theorem 3 and Proposition 5 and the fact that antichains in the well-founded graph $U \times \kappa$ are no larger than those in U .

5 Conclusion

In this paper, we have extended Ohlmann’s work [20] to the study of the memory of objectives. We have introduced different variants of well-monotone universal graphs adequate to the various models of memory appearing in the literature, and we have characterised the memory of objectives through the existence of such universal graphs (Theorems 3 and 4).

Possible applications. We expect these results to have two types of applications. The first one is helping to find tight bounds for the memory of different families of objectives. We have illustrated this use of universal graphs by providing non-trivial tight bounds on the memory of some concrete examples. In the full version [7, Section 4], we further recover known results about the memory of topologically closed objectives [8] and Muller objectives [12]. While finding universal graphs and proving their correctness might be difficult, we believe that they are a useful support to guide our intuition, and they provide a standardised method to formalise proofs of upper bounds on memory requirements.

The second kind of application discussed in the paper is the study of combinations of objectives. We have used our characterizations to bound the memory requirements of finite lexicographical product of objectives (Section 4). We have also established that intersections of objectives with finite ε -memory always have locally finite ε -free memory. Finally, we have proved that prefix-independent Σ_2^0 objectives with finite ε -memory are closed under countable unions. We believe that the new angle offered by universal graphs will help to better understand general closure properties of memory.

Open questions. Many questions remain open. First of all, as discussed in Section 4, we have proved that objectives admitting universal monotone wqo’s are closed by intersection. However, we do not know whether the larger class of objectives with unbounded finite ε -free memory is closed under intersection (Conjecture 12). A related question is therefore understanding what are exactly the objectives admitting universal monotone wqo’s.

In the realm of positional objectives, a long-lasting open question is Kopczyński’s conjecture [15]: are unions of prefix-independent positional objectives positional? This conjecture has recently been disproved for finite game graphs by Kozachinskiy [16], but it remains open for arbitrary game graphs. We propose a generalisation of Kopczyński’s conjecture in the case of ε -memory.

► **Conjecture 15.** *Let $W_1 \subseteq C^\omega$ and $W_2 \subseteq C^\omega$ be two prefix-independent objectives with ε -memory $\leq n_1, n_2$, respectively. Then $W_1 \cup W_2$ has ε -memory $\leq n_1 n_2$.*

Objectives that are ω -regular (those recognised by a deterministic parity automaton, or, equivalently, by a non-deterministic Büchi automaton) have received a great deal of attention over the years. However, very little is known about their memory requirements, and even about their positionality. By now, thanks to a recent work of Bouyer, Casares, Randour and Vandenhove [1], which relies on Ohlmann’s characterisation, positionality is understood for objectives recognised by deterministic Büchi automata.

Characterising positionality or memory requirements for other general classes of ω -regular objectives, such as those recognised by deterministic co-Büchi automata or by deterministic automata of higher parity index remains an open and challenging endeavour. Similarly, one may turn to (non-necessarily ω -regular) objectives with topological properties, for instance, it is not known by now which topologically open objectives (or, recognised by infinite deterministic reachability automata) are positional, or finite memory. We hope that the newly available tools presented in this paper will help progress in this direction.

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122:18 Characterising Memory in Infinite Games

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