A positional Π_3^0-complete objective

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Abstract

We study zero-sum turn-based games on graphs. In this note, we show the existence of a game objective that is $\mathbf{\Pi}_3^0$-complete for the Borel hierarchy and that is *positional*, i.e., for which positional strategies suffice for the first player to win over arenas of arbitrary cardinality. To the best of our knowledge, this is the first known such objective; all previously known positional objectives are in $\mathbf{\Sigma}_3^0$. The objective in question is a qualitative variant of the well-studied *total-payoff objective*, where the goal is to maximise the sum of weights.

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1 Context and contribution

We consider infinite-duration zero-sum two-player games played on (potentially infinite) graphs [6]. In this context, two players, Eve and Adam, take turns in moving a token along the edges of an edge-labelled directed game graph. This interaction produces an infinite path inducing an infinite word x in C^ω, where C is the (potentially infinite, but assumed countable) alphabet of labels. An *objective* $W \subseteq C^\omega$ is specified in advance; Eve wins the game if the word x belongs to W, otherwise Adam wins.

A strategy is called *positional* (or memoryless) if it chooses the next move only according to the current vertex of the graph containing the token, regardless of past moves. An objective is called *positional*[1] if for any game graph (of arbitrary cardinality), Eve has a positional strategy winning from every vertex from which she has a winning strategy.

To the best of our knowledge, all previously known positional objectives belong to $\mathbf{\Sigma}_3^0$, the (existential) third level of the Borel hierarchy in the usual product topology[2] over C^ω. For instance, the positionality of the ω-regular languages is well-understood [4], but they all lie in $\mathbf{\Delta}_3^0 = \mathbf{\Sigma}_3^0 \cap \mathbf{\Pi}_3^0$ (as shown in [3]). There are additional examples stemming for characterizations for objectives in $\mathbf{\Sigma}_1^0$, $\mathbf{\Pi}_1^0$, and $\mathbf{\Sigma}_2^0$ (see, respectively, [2], [5] and [10]). The following natural $\mathbf{\Sigma}_3^0$-complete objective is also shown to be positional in [6]: $\mathrm{InfOcc} = \{x \in \mathbb{N}^\omega \mid \exists c \in \mathbb{N}, |x|_c = \infty\}$, where $|x|_c$ denotes the number of occurrences of c in x (InfOcc is thus the set of words in which some number occurs infinitely often). However, its complement, which is a $\mathbf{\Pi}_3^0$-complete objective, is not positional — to see it, consider a game graph with a single vertex where Eve has to choose among infinitely many self-loops, each labeled with a different number

[1] The literature sometimes uses "half-positional" for this notion, since there is a requirement on Eve's strategy complexity, but not on Adam's.

[2] We recall that the open sets of this topology are those of the form LC^ω, for $L \subseteq C^*$ a set of finite words.

$c \in \mathbb{N}$. This leads to the following question: does there exist a positional objective that does not belong to $\boldsymbol{\Sigma}_3^0$?

We answer this question positively, by showing that the following $\boldsymbol{\Pi}_3^0$-complete objective over $C = \mathbb{Z}$ is positional:

$$\mathrm{SumToInfinity} = \{w_0 w_1 \ldots \in \mathbb{Z}^\omega \mid \lim_{k\to\infty} \sum_{i=0}^{k-1} w_i = +\infty\}.$$

This objective is a qualitative variant of a *total-payoff objective* (also called *total-reward objective*), where the goal is to maximize the (lim sup or lim inf of the) sum of weights. Total-payoff objectives are positional over *finite* arenas [7]. However, over infinite arenas, they are in general not positional, with various classes of strategies needed depending on the variant considered (lim sup or lim inf, and with a rational, $+\infty$, or $-\infty$ threshold) and the class of arenas [10, 1]. Objective SumToInfinity is a specific natural variant which turns out to have a remarkably low strategy complexity even over the most general class of arenas. Our results fill a gap in the understanding of quantitative objectives [1].

Note that SumToInfinity is prefix-independent (i.e., for all $x \in \mathbb{Z}^*$ and $x' \in \mathbb{Z}^\omega$, $x' \in \mathrm{SumToInfinity}$ if and only if $xx' \in \mathrm{SumToInfinity}$).

▶ **Theorem 1.** *The objective* SumToInfinity *is* $\boldsymbol{\Pi}_3^0$*-complete and positional.*

The rest of the note is devoted to the proof of Theorem 1. We quickly show in Section 2 that SumToInfinity is $\boldsymbol{\Pi}_3^0$-complete; our main contribution, in Section 3, is a positionality proof based on constructing *(almost)-universal graphs* for SumToInfinity, and applying [9, Theorem 3.2].

Naturally, a follow-up open question is whether every level of the Borel hierarchy admits a complete objective that is positional.

2 $\boldsymbol{\Pi}_3^0$-completeness of SumToInfinity

We refer to [8] for definitions on the Borel hierarchy.

To show that SumToInfinity is in $\boldsymbol{\Pi}_3^0$, observe that

$$\mathrm{SumToInfinity} = \bigcap_{n\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \bigcap_{k\geq j} \{w_0 w_1 \ldots \in \mathbb{Z}^\omega \mid \sum_{i=0}^{k-1} w_i \geq n\},$$

where the inner sets $\{w_0 w_1 \ldots \in \mathbb{Z}^\omega \mid \sum_{i=0}^{k-1} w_i \geq n\}$ are clopen.

To show that SumToInfinity is $\boldsymbol{\Pi}_3^0$-hard, we reduce the following $\boldsymbol{\Pi}_3^0$-hard objective [8, Ex. 23.2] to it:

$$\mathrm{FinOcc} = \{x \in \mathbb{N}^\omega \mid \forall c \in \mathbb{N}, |x|_c \text{ is finite}\}.$$

We recall that for a reduction, we need to show a continuous mapping $f\colon \mathbb{N}^\omega \to \mathbb{Z}^\omega$ such that $f^{-1}(\mathrm{SumToInfinity}) = \mathrm{FinOcc}$. Such a mapping is defined by:

$$f(c_0 c_1 \ldots) = w_0 w_1 \ldots, \quad \text{with } w_i = c_{i+1} - c_i.$$

The function f is continuous, as if $x, x' \in \mathbb{N}^\omega$ are two words with a common prefix of size k, then $f(x)$ and $f(x')$ have a common prefix of size $k-1$.

Let us show that $f^{-1}(\mathrm{SumToInfinity}) = \mathrm{FinOcc}$. Note that $\sum_{i=0}^{k-1}(c_{i+1} - c_i) = c_k - c_0$. If $c_0 c_1 \ldots \notin \mathrm{FinOcc}$, then $c_k - c_0$ takes infinitely often a constant value $c \in \mathbb{N}$. Therefore, $\sum_{i=0}^{k-1}(c_{i+1} - c_i) \not\to +\infty$. Conversely, if $c_0 c_1 \ldots \in \mathrm{FinOcc}$, then, for all $c \in \mathbb{N}$, $c_k - c_0 > c$ for all sufficiently large k, so $\sum_{i=0}^{k-1}(c_{i+1} - c_i) \to +\infty$.

3 Positionality of SumToInfinity

To keep this note short, we include only the crucial formal definitions; we refer the reader to [9] for additional context.

In what follows, the word *graph* stands for a directed graph with edges labelled by elements of C. The set of vertices of a graph G is written $V(G)$, and its edges $E(G) \subseteq V(G) \times C \times V(G)$.

Almost-universality. A *tree* is a graph with a root t_0 such that all vertices admit a unique path from t_0. A graph morphism from G to H is a map $f\colon V(G) \to V(H)$ such that for each edge $v \xrightarrow{c} v'$ in G, $f(v) \xrightarrow{c} f(v')$ is an edge in H. We write $G \to H$ if there exists such a morphism. A *well-ordered graph* is a graph whose vertices are well-ordered by an ordering $\leq$. An ordered graph is *monotone* if $u \geq v \xrightarrow{c} v' \geq u'$ implies $u \xrightarrow{c} u'$. A graph *satisfies* an objective W if the labelling of any infinite path on it belongs to W. For a prefix-independent objective W and a cardinal κ, a graph U if said to be (κ, W)-*almost-universal* if

- U satisfies W, and
- for all trees T of size $< \kappa$ satisfying W, there is a vertex v_0 such that $T[v_0] \to U$,

where $T[v_0]$ is the restriction of T to vertices reachable from v_0. We will rely on the following result:

▶ **Lemma 2** ([9], Theorem 3.2 and Lemma 4.5)**.** *Let W be a prefix-independent objective. If, for all cardinal κ, there exists a well-ordered monotone (κ, W)-almost-universal graph, then W is positional.*

Objective SumToInfinity is prefix-independent. To prove that it is positional, it therefore suffices, for every cardinal κ, to build a $(\kappa, \mathrm{SumToInfinity})$-almost-universal graph U. In what follows, let $C = \mathbb{Z}$ and let κ be a cardinal.

Definition of U. We will manipulate finite tuples of ordinals. For such a tuple u, we write $u_{<i}$ for the restriction of u to its first i coordinates:

$$(u_0, \ldots, u_n)_{<i} = (u_0, \ldots, u_{i-1}).$$

We let $|u|$ denote the length of u, for instance $|(0,1)| = 2$. Recall the lexicographic ordering:

$$u >_{\mathrm{lex}} u' \iff u' \text{ is a prefix of } u \text{ or } \exists i, [u_{<i} = u'_{<i} \text{ and } u_i > u'_i].$$

Consider the graph U defined over $V(U) = \bigcup_{n<\omega} \kappa^n$ by

$$E(U) = \{u \xrightarrow{w} u' \mid |u| + w \geq |u'| \text{ and } [|u| + w = |u'| \implies u >_{\mathrm{lex}} u']\}.$$

Intuitively, the length of the tuples in U encodes an underapproximation of the sum of weights in a given path: an edge either tracks precisely the sum of weights (when $|u| + w = |u'|$), or it underestimates it (when $|u| + w > |u'|$). In the former case, there is an additional requirement on u', which is that it decreases for $<_{\mathrm{lex}}$. In the latter case, the tuple can have any value. These rules prevent in particular the existence of cycles with sum of weight 0 in U; to go back to the same vertex, some underestimating is necessary.

The order over U is then defined by

$$u > u' \iff |u| > |u'| \text{ or } [|u| = |u'| \text{ and } u >_{\mathrm{lex}} u'].$$

We raise the reader's attention on the fact that the order over U does not coincide with the lexicographic order: for instance, $(0,0) > (1)$ in U.

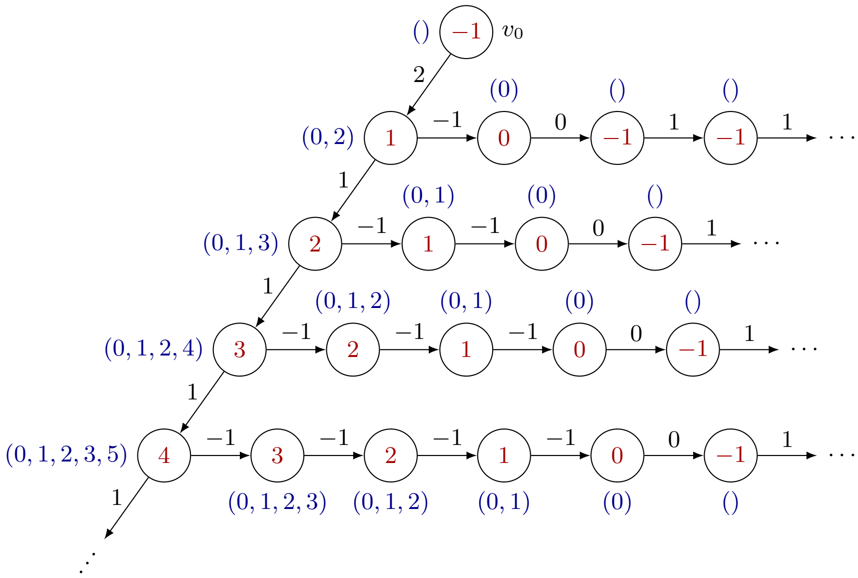

Figure 1 Tree T used in Example 4. Remember that the weights (in black) label edges. This tree satisfies SumToInfinity (as any infinite path ends with 1^ω). The value in red inside each vertex is the value $n(\cdot)$ defined in the proof of Lemma 7. The top vertex v_0 is such that $n(v_0) = -1 < 0$, so we can assume it is the vertex given by Claim 7.1. Every path from v_0 not reaching another vertex with value -1 is tight. Observe that there is exactly one infinite tight path from v_0 (staying on the left branch), indeed satisfying the property of Claim 7.2. The tuples in blue next to vertices correspond to the morphism to U built in the proof of Claim 7.3.

▶ **Lemma 3.** *The graph $(U, \leq)$ is a well-ordered monotone graph.*

Proof. It is immediate that the order over U is well-founded and total. Let us check that U is monotone. Let $u \geq v \xrightarrow{w} v' \geq u'$ in U. Then, $|u| + w \geq |v| + w \geq |v'| \geq |u'|$. If one of these inequalities is strict, then, $|u| + w > |u'|$. Otherwise, $|u| + w = |u'|$ and $u \geq_{\text{lex}} v >_{\text{lex}} v' \geq_{\text{lex}} u'$. We conclude that $u \xrightarrow{w} u'$. ◀

▶ **Example 4.** Before proving the $(\kappa, \text{SumToInfinity})$-almost-universality of U, we give one example of a morphism of a tree into U. We consider the tree T from Figure 1. The blue tuples next to each vertex indicate a possible morphism from T to U. The morphism given is exactly the one built by our proof of almost-universality below; we incite the reader to come back to this example as an illustration of the upcoming proof. ⌟

Almost-universality of U. We now prove the following.

▶ **Theorem 5.** *The graph U is $(\kappa, \text{SumToInfinity})$-almost-universal.*

We prove the two conditions for almost-universality in two separate lemmas.

▶ **Lemma 6.** *The graph U satisfies* SumToInfinity.

Proof. Take an infinite path $u^0 \xrightarrow{w_0} u^1 \xrightarrow{w_1} \dots$ in U. For all i, let

$$b_i = \begin{cases} 0 & \text{if } |u^i| + w_i = |u^{i+1}|, \\ 1 & \text{otherwise.} \end{cases}$$

For each i, we have $|u^i| + w_i \geq |u^{i+1}| + b_i$. Therefore, for all k,

$$\sum_{i=0}^{k-1} w_i \geq |u^k| - |u^0| + \sum_{i=0}^{k-1} b_i \geq \sum_{i=0}^{k-1} b_i - |u^0|.$$

If $\sum_{i=0}^{k-1} b_i$ goes to $+\infty$, then by the above it also holds that $\sum_{i=0}^{k-1} w_i \to +\infty$, as wanted. So we assume otherwise: there is i_0 such that for $i \geq i_0$, $b_i = 0$. Then, for all $i \geq i_0$, we have $|u^i| + w_i = |u^{i+1}|$ and thus $u^i >_{\text{lex}} u^{i+1}$; this contradicts the well-foundedness of $<_{\text{lex}}$. ◀

▶ **Lemma 7.** *For all trees $T < \kappa$ satisfying* SumToInfinity*, there exists a vertex v_0 of T such that $T[v_0] \to U$.*

Proof. Let $T < \kappa$ be a tree satisfying SumToInfinity. Given a finite path π, we let $w(\pi)$ denote the sum of the weights appearing on π. For all $v \in T$, define

$$n(v) = -\inf\{w(\pi) \mid \pi \text{ is a non-empty finite path from } v\} \in \mathbb{Z} \cup \{+\infty\}.$$

We note that for all edges $v \xrightarrow{w} v'$, it holds that $n(v) + w \geq n(v')$.

▷ Claim 7.1. There exists a vertex v_0 such that $n(v_0) < 0$.

Proof. Assume towards a contradiction that for all vertices v, $n(v) \geq 0$. In other words, from all vertices, there is a non-empty path of weight ≤ 0. By concatenating such paths, we get a path whose weight does not converge to $+\infty$, which contradicts the fact that T satisfies SumToInfinity. ◁

Using the above claim, let v_0 be a vertex such that $n(v_0) < 0$. We will construct a mapping $\phi \colon T[v_0] \to U$. The following claim will be useful for the definition of this morphism. We say that an edge $v \xrightarrow{w} v'$ of T is *tight* if $n(v) + w = n(v')$, and that a (finite or infinite) path is *tight* if it is comprised only of tight edges.

▷ Claim 7.2. Let π be an infinite tight path from v_0. For each $k \geq 0$, there are finitely many vertices v on π satisfying $n(v) \leq k$.

Proof. Denote $\pi = v_0 \xrightarrow{w_0} v_1 \xrightarrow{w_1} \dots$; since all edges are tight, we have

$$n(v_i) = n(v_0) + \sum_{j<i} w_j.$$

Since π satisfies SumToInfinity, $\sum_{j<i} w_j$ converges to $+\infty$; therefore, so does $n(v_i)$. The result follows. ◁

We now define a morphism $\phi \colon T[v_0] \to U$. First, notice that all vertices v in $T[v_0]$ are such that $n(v) < +\infty$; otherwise, we would also have $n(v_0) = +\infty$. For v in $T[v_0]$, we define the length of the tuple $\phi(v)$ to be $\max\{n(v) + 1, 0\} \in \mathbb{N}$ (in particular, if $n(v) < 0$, then $\phi(v)$ is the empty tuple). For a vertex v in $T[v_0]$ with $n(v) \geq 0$ and $0 \leq k \leq n(v)$, the k-th coordinate of $\phi(v)$ is defined as follows. Informally, we count the number of vertices v' with $n(v') \leq k$ on a tight path starting in v. Formally, define $T_{v,k}$ to be the graph with vertices

$$V(T_{v,k}) = \{v' \in V(T) \mid \text{there is a tight path from } v \text{ to } v' \text{ and } n(v') \leq k\},$$

and edges

$$E(T_{v,k}) = \{v'_1 \xrightarrow{w} v'_2 \mid w \text{ is the weight of a tight path from } v'_1 \text{ to } v'_2 \text{ whose inner vertices } v' \text{ satisfy } n(v') > k\}.$$

The graph $T_{v,k}$ is actually a tree with root v when $k = n(v)$, but is in general a disjoint union of trees.

Claim 7.2 implies that $T_{v,k}$ is well-founded (it does not admit any infinite path). The rank of a vertex in a well-founded disjoint union of trees is an ordinal number defined to be 0 for leaves, and one plus the supremum rank of its successors for non-leaves. The rank of $T_{v,k}$, written $\mathrm{rk}(T_{v,k})$, is the supremum rank of its vertices (note that it is $< \kappa$).

We set the k-th coordinate of $\phi(v)$ to be the rank of $T_{v,k}$, thus

$$\phi(v) = (\mathrm{rk}(T_{v,0}), \mathrm{rk}(T_{v,1}), \ldots, \mathrm{rk}(T_{v,n(v)})).$$

▷ Claim 7.3. The map $\phi\colon V(T[v_0]) \to V(U)$ defines a morphism from $T[v_0]$ to U.

Proof. Consider an edge $v \xrightarrow{w} v'$; we show that $\phi(v) \xrightarrow{w} \phi(v')$ is an edge in U.

First, notice that we have in general that $n(v) + w \geq 0$: indeed, $\pi = v \xrightarrow{w} v'$ is a non-empty path from v so, by definition of $n(v)$, $-n(v) \leq w$.

If $n(v') < 0$, then

$$|\phi(v)| + w = \max(n(v) + 1, 0) + w \geq n(v) + 1 + w \geq 1 > 0 = |\phi(v')|,$$

thus $\phi(v) \xrightarrow{w} \phi(v')$ is an edge in U.

We now assume in the rest of the proof that $n(v') \geq 0$. We reason according to whether the edge $v \xrightarrow{w} v'$ is tight.

- First, assume that edge $v \xrightarrow{w} v'$ is not tight, i.e., that $n(v) + w > n(v')$. Then the argument is similar to the previous one:

  $$|\phi(v)| + w \geq n(v) + 1 + w > n(v') + 1 = |\phi(v')|,$$

  where the last equality uses that $n(v') \geq 0$.
- Second, assume $v \xrightarrow{w} v'$ is tight, i.e., $n(v) + w = n(v')$. Therefore,

  $$|\phi(v)| + w \geq n(v) + 1 + w = n(v') + 1 = |\phi(v')|,$$

  so it suffices to show that $\phi(v) >_{\mathrm{lex}} \phi(v')$.

  Let $k \leq \min\{n(v), n(v')\}$. As v' is reachable from v, $T_{v',k}$ is a subgraph of $T_{v,k}$. Therefore, $\mathrm{rk}(T_{v,k}) \geq \mathrm{rk}(T_{v',k})$. If $n(v) > n(v')$, we deduce that $\phi(v) >_{\mathrm{lex}} \phi(v')$ ($\phi(v)$ is a longer tuple and starts with values at least as large). If $n(v) \leq n(v')$, with $k = n(v)$, since v is a vertex (in fact, the root) of $T_{v,n(v)}$ but not of $T_{v',n(v)}$, we get $\mathrm{rk}(T_{v,n(v)}) > \mathrm{rk}(T_{v',n(v)})$. We also conclude that $\phi(v) >_{\mathrm{lex}} \phi(v')$, as required. ◁

This ends the proof of positionality of SumToInfinity. ◀

References

1 Sougata Bose, Rasmus Ibsen-Jensen, David Purser, Patrick Totzke, and Pierre Vandenhove. The power of counting steps in quantitative games. In *CONCUR 2024*, volume 311 of *LIPIcs*, pages 13:1–13:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPIcs.CONCUR.2024.13.

2 Patricia Bouyer, Nathanaël Fijalkow, Mickael Randour, and Pierre Vandenhove. How to play optimally for regular objectives? In *ICALP 2023*, volume 261 of *LIPIcs*, pages 118:1–118:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2023. doi:10.4230/LIPICS.ICALP.2023.118.

3 J. Richard Büchi and Lawrence H. Landweber. Definability in the monadic second-order theory of successor. *J. Symb. Log.*, 34(2):166–170, 1969. doi:10.2307/2271090.

4 Antonio Casares and Pierre Ohlmann. Positional ω-regular languages. In *LICS 2024*, pages 21:1–21:14. ACM, 2024. doi:10.1145/3661814.3662087.

5 Thomas Colcombet, Nathanaël Fijalkow, and Florian Horn. Playing safe, ten years later. *Log. Methods Comput. Sci.*, 20(1), 2024. doi:10.46298/LMCS-20(1:10)2024.

6 Nathanaël Fijalkow, Nathalie Bertrand, Patricia Bouyer-Decitre, Romain Brenguier, Arnaud Carayol, John Fearnley, Hugo Gimbert, Florian Horn, Rasmus Ibsen-Jensen, Nicolas Markey, Benjamin Monmege, Petr Novotný, Mickael Randour, Ocan Sankur, Sylvain Schmitz, Olivier Serre, and Mateusz Skomra. *Games on Graphs*. Cambridge University Press, 2024. To Appear.

7 Hugo Gimbert and Wiesław Zielonka. When can you play positionally? In *MFCS 2004*, volume 3153 of *Lecture Notes in Computer Science*, pages 686–697. Springer, 2004. doi:10.1007/978-3-540-28629-5_53.

8 Alexander Kechris. *Classical Descriptive Set Theory*. Graduate Texts in Mathematics. Springer New York, 1995. doi:10.1007/978-1-4612-4190-4.

9 Pierre Ohlmann. Characterizing Positionality in Games of Infinite Duration over Infinite Graphs. *TheoretiCS*, Volume 2, January 2023. doi:10.46298/theoretics.23.3.

10 Pierre Ohlmann and Michał Skrzypczak. Positionality in Σ^0_2 and a completeness result. In *STACS 2024*, volume 289 of *LIPIcs*, pages 54:1–54:18. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2024. doi:10.4230/LIPICS.STACS.2024.54.